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Method of fundamental solutions with regularization techniques for Cauchy problems of elliptic operators

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Abstract

In this paper we combine the method of fundamental solutions with various regularization techniques to solve Cauchy problems of elliptic differential operators. The main idea is to approximate the unknown solution by a linear combination of fundamental solutions whose singularities are located outside the solution domain. To solve effectively the discrete ill-posed resultant matrix, we use three regularization strategies under three different choices for the regularization parameter. Several examples on problems with smooth and non-smooth geometries in 2D and 3D spaces using under-, equally, and over-specified Cauchy data on an accessible boundary are given. Numerical results indicate that the generalized cross-validation and *L*-curve choice rulers for Tikhonov regularization and damped singular value decomposition strategy are most effective when using the same numbers of collocation and source points. It has also been observed that the use of more Cauchy data will greatly improve the accuracy of the approximate solution.

Keywords: Method of fundamental solutions; Cauchy problems; Inverse problems; Regularization methods

1. Introduction

The method of fundamental solutions (MFS) has recently been used extensively for solving various types of linear partial differential equations. For instances, Laplace equation [1–3], biharmonic [4,5], elastostatics problems [6], and wave scattering problems [7,8]. More recently, the MFS has successfully been applied to approximate the solutions of non-homogeneous linear and nonlinear Poisson equations [9–12]. Details can be found in the review papers of Fairweather and Karageorghis [13] and Golberg and Chen [14]. These problems are called well-posed direct problems in which the Dirichlet or Neumann data on the whole boundary are known. In the studies of inverse problems, the boundary data are given with noises on part of the accessible boundary. This usually poses difficulty on most of the traditional numerical methods to obtaining acceptable numerical approximation to the solution. The truly meshless MFS is an excellent candidate for solving these kinds of inverse problems.

The Cauchy problem for an elliptic equation is a typical ill-posed problem whose solution does not depend continuously on the boundary data. That is, a small error in the specified data may result in an enormous error in the numerical solution. The Cauchy problem for the Laplace equation arises from many branches of science and engineering such as non-destructive testing [15], steady-state inverse heat conduction [16], and electro-cardiology [17]. Some numerical methods have recently been developed [18–24]. To handle the severe ill-conditioning problem, the truncated singular value decomposition (TSVD) and first order Tikhonov regularization (TR) techniques with *L*-curve choice criterion have been used in

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[25] and [26,27], respectively, to solve Cauchy problem of Helmholtz-type equations with equally or over-specified Cauchy data.

In this paper, we combine the MPS with various regularization techniques to solve Cauchy problems of Laplace and Helmholtz-type equations using under-, equally, and over-specified Cauchy data. This provides an effective and stable numerical approximation to the solution. The outline of the paper is as follows: the formulation of Cauchy problem is first given in Section 2. The key idea of MFS is then reviewed in Section 3. In Section 4, we combine the MFS with the following regularization techniques:

- Tikhonov regularization (TR) method,
- damped singular value decomposition (DSVD),
- truncated singular value decomposition (TSVD)

under three *choice rules for regularization parameters*:

- discrepancy principle (DP),
- L-curve criterion (LC),
- generalized cross-validation (GCV),

for the numerical approximation of the solution to Cauchy problems of elliptic operators. Details on these regularization techniques can be found in [28]. Numerical examples and comparisons given in Section 5 indicate that the TR and DSVD regularization techniques combined with the GCV or the LC choice rules for the regularization parameter are most effective for the cases of resultant square matrix, i.e. the total number of collocation points equals the total number of source points. For non-square matrix case, the LC can still provide an acceptable approximation but the GCV does not work well. The TSVD under the three choice rules for the regularization parameter on under-specified boundary data has found to be very unusable. This is different from the conclusion presented in [25] where the Cauchy data is given on $\frac{3}{4}$ part of the whole boundary. Furthermore, our numerical experiments show that the DP works well for Laplace and modified Helmholtz equation but fails for Helmholtz equation in which the resultant matrix is complex. The DP rule needs an a priori noise level which in practice is difficult to be determined. For this reason, we focus in this paper the use of noise-free choice rules, i.e. LC and GCV. It is also observed that the use of over-specified Cauchy data will greatly improve the accuracy of an approximate solution. Finally, the conclusion is given in Section 6.

2. Formulation of the Cauchy problem

Let Ω be a bounded and simply connected domain in $\mathscr{R}^d$, $d = 2, 3$ with Lipschitzian boundary and Γ is a portion of the boundary $\partial\Omega$. A Cauchy problem for an elliptic equation in multidimensional domain is to determine a distribution function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that satisfies

$$Lu = 0 \quad \text{in } \Omega, \tag{1}$$

$$u|_\Gamma = \varphi, \tag{2}$$

$$\left.\frac{\partial u}{\partial n}\right|_\Gamma = \psi, \tag{3}$$

where L is a second order elliptic operator in $\mathscr{R}^d$, φ and ψ are, respectively, the Dirichlet and Neumann data specified on boundary Γ, $\partial u/\partial n$ is the outward normal derivative of u at Γ. It is noted that if Γ is the whole boundary $\partial\Omega$, the problem is a well-posed direct problem. In this paper, we consider three elliptic operators: Laplace ($Lu = \Delta u$), Helmholtz ($Lu = \Delta u + k^2 u$), and modified Helmholtz ($Lu = \Delta u - k^2 u$), where Δ is the Laplace operator in $\mathscr{R}^d$ and k is a real positive constant. See Fig. 1 for an illustration of the problem setting.

Problem (1)–(3) when $L = \Delta$ is a classical Cauchy problem for Laplace equation which is typically ill-posed [29]. Under an additional a priori condition, a continuous dependence of the solution on the given data can be obtained [30,31]. The instability of the solution to the Cauchy problem of Helmholtz equation can be shown by considering the following example in 2D case:

let

$$\Omega = \{P = (x, y) | 0 < x < 1, 0 < y < 1\} \tag{4}$$

and

$$\Gamma = \partial\Omega \cap \{y = 0\}.$$

The solution when

$$\varphi = \frac{1}{k^2 n^2}\sin(\sqrt{n^2+1}kx), \quad \psi = -\frac{1}{kn}\sin(\sqrt{n^2+1}kx)$$

is given by

$$u(x, y) = \frac{1}{k^2 n^2}\sin(\sqrt{n^2+1}kx)e^{kny},$$

for $(x, y) \in \Omega$. Therefore, we have

$$\sup_{P\in\Gamma}\{|\varphi| + |\psi|\} = O\left(\frac{1}{n}\right) \to 0 \quad \text{as } n \to \infty,$$

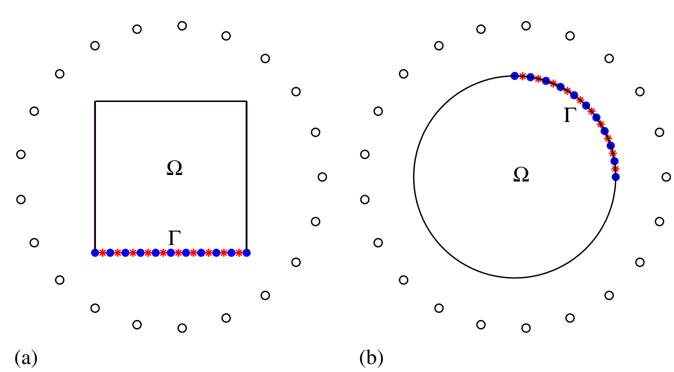

Fig. 1. Cauchy problem for two domains Ω. Dots (•) and (∗) are collocation points for Dirichlet and Neumann data, respectively. Circles (○) represent source points. (a) Square domain and (b) circle domain.