Engineering Analysis with Boundary Elements 31 (2007) 577–585

ENGINEERING ANALYSIS with BOUNDARY ELEMENTS

www.elsevier.com/locate/enganabound

Exact explicit time integration of hyperbolic partial differential equations with mesh free radial basis functions

E.J. Kansa

Department of Mechanical and Aeronautical Engineering, University of California, Davis 95616-5294, USA

Received 22 August 2006; accepted 13 December 2006
Available online 23 February 2007

Abstract

This study is a progress report that examines the numerical solution of inviscid hyperbolic partial differential equations (PDEs) without the need for upwind differencing and other numerical artifacts. The fixed frame PDEs are locally transformed by rotating and translating the coordinate system at each local discretization point. These transformations yield a simpler PDE system that is effectively linearized. It is assumed that in this transformed local frame within a time interval, Δt, the dependent variables are products of the spatial dependent radial basis functions (RBFs), and the time dependent expansion coefficients, $\boldsymbol{\chi}(t)$. This linearization is exploited by transforming the PDEs into systems of linear ordinary differential equations (ODEs) in terms of the expansion coefficients. The affine space decomposition is used to obtain an ODE system of N_i ODEs in N_i unknowns that can be integrated exactly in time. Then the entire set of N expansion coefficients is found. Numerical results show that hyperbolic PDEs can be integrated in time without upwinding and the root mean square errors between the exact and numerical solutions are indeed very small.

Keywords: Hyperbolic PDEs; Radial basis functions; Linearized ODE systems; Moving data centers; Exact explicit time solutions

1. Introduction

The interest in mesh-free methods to solve partial differential equations (PDEs) has grown considerably in the past 15 years. The two principal reasons are: (1) mesh generation over two- and three-dimensional complicated domains may require weeks or months to produce a well-behaved mesh, and (2) the convergence rate of traditional methods are typically second order, requiring very fine discretization. The fine discretization required may require more operations than mesh-free methods, even though the traditional methods are compactly supported. The mesh-free radial basis functions (RBFs) have been shown to be particularly attractive by Fedoseyev et al. [1] and Cheng et al. [2] because of the exponential convergence of certain C^∞ RBFs that has been observed. Various RBFs have been successfully applied to obtain very accurate and efficient solutions to PDEs of engineering interest [3,4].

One of the most used RBFs is the multiquadric (MQ) RBF. The generalized MQ basis function, $\phi_j(\mathbf{x}) = [(\mathbf{x}-\mathbf{x}_j)^2 + c_j^2]^\beta$, where $\mathbf{x}, \mathbf{x}_\ell \in \Re^d$. Commonly used values for β are $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$, although various other exponents have been successfully used. The impetus in RBF research arose from the paper by Franke [5] in 1982. Madych and Nelson [6] and Madych [7] have proven theoretically that MQ interpolation converges exponentially as $\eta^{c/h}$, where η is a real number, $\eta < 1$, and h is the average data center separation. The application of RBFs in the numerical solution of PDEs was demonstrated in [8] using the asymmetric collocation approach.

PDEs can be represented in the fixed Euler frame, a moving frame such as the Lagrangian formulation, or a frame moving with the characteristic velocities. In this study, a moving frame representation is chosen so the dependent variables can be assumed to be separable in space and time, where the spatial dependence arises from the choice of basis function and the time dependence arises from the expansion coefficients, $\boldsymbol{\chi}(t)$. However, the spatial basis functions are implicit functions of time because data

E-mail address: ejkansa@ucdavis.edu.

0955-7997/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.enganabound.2006.12.001

centers can move in time. Depending upon the problem description, some physical boundary loci may either be fixed in time or move. Internal boundaries representing propagating discontinuities move in time. Because the points can move randomly, a scattered data method is needed; and MQ-RBFs were shown by Franke [5] to perform optimally. For hyperbolic PDEs, it is convenient to decompose a domain, $\Omega = \cup_\sigma \Omega_\sigma$ where Ω_σ is a subdomain whose boundaries are either the physical boundaries and/or internal boundaries such as discontinuities.

In the domain, Ω_σ, the spatial dependency of the dependent variables, U^k, between physical and internal discontinuous boundaries are represented by a combination of continuous two-dimensional exponentially convergent RBFs and polynomials. The discontinuities are curves that are products of polynomials or splines in the tangential directions and Heaviside functions in the normal propagating direction. One can construct a generalized Heaviside function, $H_\sigma(\mathbf{x})$, that is unity inside Ω_σ, but zero outside. If a wave breaks forming a discontinuity, then Ω_σ will be partitioned into two or more subdomains. The starting point for the RBF–PDE scheme is interpolation, since the initial value problem is essentially an interpolation problem.

The expansion coefficients, $\{\boldsymbol{\chi}^k(t=0)\}$, for the kth dependent variable, $\mathbf{U}^k(\mathbf{x},t)$ are found by specifying the initial conditions of the dependent variables. However, these expansions have no conservation constraints. For time dependent PDE problems in which the dependent variables are continuous over Ω_σ, it is desirable to have polynomial reproduction, strict conservation, and represent the spatially dependent variables as a linear combination of RBFs. The matrix Φ consists of all $\phi_\ell(\mathbf{x}_j)$, where $\ell j \in [1,\ldots,N_i,N_{i+1},\ldots,N]$. There are N data centers discretizing Ω_σ, with N_i data centers in $\Omega_\sigma\backslash\partial\Omega_\sigma$ and N_b data centers along $\partial\Omega_\sigma$, such that $N = N_i + N_b$. For polynomial reproduction, a set of polynomials, $\mathbf{P}$, excluding constants is appended to the expansion. It is assumed that set of dependent variable, $\mathbf{U}^k$, can be integrated over Ω_σ to obtain the extensive quantities such as mass, momentum components, and total energy contained in the volume, Ω_σ. The integrals of mass, momentum components, and total energy to form the conservative quantity, $\Theta^k = \int_{\Omega_\sigma} \mathbf{U}^k\, dx$.

It is convenient to partition the contributions from interior and boundary points to form block matrices. Define an RBF matrix $\boldsymbol{\Phi}_{ii}$ that consists of points, $\mathbf{x}_\ell$ and $\mathbf{x}_j \in \Omega_\sigma\backslash\partial\Omega$; define an RBF matrix $\boldsymbol{\Phi}_{ib}$ that consists of points, $\mathbf{x}_\ell \in \Omega_\sigma\backslash\partial\Omega_\sigma$ and $\mathbf{x}_j \in \partial\Omega_\sigma$; define a RBF matrix $\boldsymbol{\Phi}_{bi}$ that consists of $\mathbf{x}_\ell \in \partial\Omega_\sigma$ and $\mathbf{x}_j \in \Omega_\sigma\backslash\partial\Omega$; and an RBF matrix $\boldsymbol{\Phi}_{bb}$ that consists of $\mathbf{x}_\ell \in \partial\Omega_\sigma$ and $\mathbf{x}_j \in \partial\Omega_\sigma$. Define $\mathbf{1}_i$ to be a column vector of N_i ones, and $\mathbf{1}_b$ to be a column vector of N_b ones; both are used for conservation constraints. Define $\mathbf{P}_i$ to be an $N_i \times m$ polynomial matrix over the interior, and $\mathbf{P}_b$ is an $N_b \times m$ polynomial matrix on the boundary; both are used for polynomial reproducibility. The other entries are the conservation enforcing row matrices, $\boldsymbol{\Psi}_i(\mathbf{x}_j) = \int_{\Omega_\sigma} \phi(\mathbf{x}-\mathbf{x}_j)\,d\mathbf{x}, \mathbf{x}_j \in \Omega_\sigma\backslash\partial\Omega_\sigma$, and $\boldsymbol{\Psi}_b(\mathbf{x}_j) = \int_{\Omega_\sigma} \phi(\mathbf{x}-\mathbf{x}_j)\,d\mathbf{x}, \mathbf{x}_j \in \partial\Omega_\sigma$. The remaining elements are: $V = \int_{\Omega_\sigma} d\mathbf{x}, \boldsymbol{\Pi} = \int_{\Omega_\sigma} \mathbf{P}(\mathbf{x})\,d\mathbf{x}$ is a $1 \times m$ matrix, and $\mathbf{0}$ is an $m \times m$ matrix of zeros. For convenience, define the matrices $\mathbf{H}_{ii}, \mathbf{H}_{ib}, \mathbf{H}_{bi}, \mathbf{H}_{bb}$, and column vectors $\boldsymbol{\chi}_i^k$. $\boldsymbol{\chi}_b^k$, $\mathbf{Z}_i^k$, and $\mathbf{Z}_b^k$ where

$$\mathbf{H}_{ii} = \begin{pmatrix} \boldsymbol{\Phi}_{ii} & \mathbf{1}_i & \mathbf{P}_i \\ \boldsymbol{\Psi}_i & V & \boldsymbol{\Pi} \\ \mathbf{P}_i^{\mathrm{T}} & \boldsymbol{\Pi}^{\mathrm{T}} & \mathbf{0} \end{pmatrix}, \tag{1}$$

$$\mathbf{H}_{ib} = \begin{pmatrix} \boldsymbol{\Phi}_{ib} & \boldsymbol{\Psi}_b & \mathbf{P}_b^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}, \tag{2}$$

$$\mathbf{H}_{bi} = \begin{pmatrix} \boldsymbol{\Phi}_{bi} & \mathbf{1}_b & \mathbf{P}_b \end{pmatrix}, \tag{3}$$

$$\mathbf{H}_{bb} = \boldsymbol{\Phi}_{bb}, \tag{4}$$

$$\boldsymbol{\chi}_i^k = \begin{pmatrix} \boldsymbol{\chi}_i^k & \boldsymbol{\chi}_{\mathrm{cons}}^k & \boldsymbol{\chi}_{\mathrm{poly}}^k \end{pmatrix}^{\mathrm{T}}, \tag{5}$$

$$\boldsymbol{\chi}_b^k = \boldsymbol{\chi}_b^k, \tag{6}$$

$$\mathbf{Z}_i^k = \begin{pmatrix} \mathbf{U}_i^k & \Theta^k & \mathbf{0} \end{pmatrix}^{\mathrm{T}}, \tag{7}$$

$$\mathbf{Z}_b^k = \mathbf{U}_b^k. \tag{8}$$

The system of equations can be compactly written as

$$\begin{pmatrix} \mathbf{H}_{ii} & \mathbf{H}_{ib} \\ \mathbf{H}_{bi} & \mathbf{H}_{bb} \end{pmatrix} \begin{bmatrix} \boldsymbol{\chi}_i^k \\ \boldsymbol{\chi}_b^k \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_i^k \\ \mathbf{Z}_b^k \end{bmatrix}, \tag{9}$$

where $\boldsymbol{\chi}^k = [\boldsymbol{\chi}_i^k, \boldsymbol{\chi}_b^k]^{\mathrm{T}}$ and $\mathbf{Z}^k = [\mathbf{Z}_i^k, \mathbf{Z}_b^k]^{\mathrm{T}}$.

To avoid ill-conditioning, both sides of the general interpolation equation are multiplied by the approximate cardinal preconditioner, $\mathbf{P}^\sigma$, for the domain, Ω_σ, see Ling et al. [9–11]. The preconditioner can be conveniently partitioned as

$$\mathbf{P}^\sigma = \begin{pmatrix} \mathbf{P}_{ii}^\sigma & \mathbf{P}_{ib}^\sigma \\ \mathbf{P}_{bi}^\sigma & \mathbf{P}_{bb}^\sigma \end{pmatrix}. \tag{10}$$

For simplicity, it shall be assumed that both $\mathbf{H}$ matrices and the right-hand sides, $\mathbf{Z}^k$, are already preconditioned for both the interpolation problem and the subsequent PDE and boundary condition approximations.

In the initial value problem, all the expansion coefficients, $\boldsymbol{\chi}^k(t=0)$, in Ω_σ, are found by specifying the initial values of $\mathbf{Z}^k(x,t=0)$, and the conservation constraint, $\Theta^k(t=0)$. Once the set of initial expansion coefficients is known, then a compatibility variable, Z^k, can be reconstructed over Ω_σ.

$$\mathbf{Z}^k(\mathbf{x},t) = \mathbf{H}(\mathbf{x})\boldsymbol{\chi}^k(t), \quad \text{for all discrete } \mathbf{x} \in \Omega_\sigma \tag{11}$$

and the local values of $U^k(\mathbf{x},t) = \mathbf{W}^{-1}\mathbf{Z}^k(\mathbf{x},t)$, locally, and $\mathbf{H}(\mathbf{x})$ is interpreted as a functional.