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Stability analysis and superconvergence for the penalty Trefftz method coupled with FEM for singularity problems

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Abstract

In this paper, we apply the effective condition number in Huang and Li [Effective condition number and superconvergence of Trefftz method coupled with high order FEM for singularity problems, Eng Anal Bound Elem 2006;30:270–83] to the penalty Trefftz methods (TMs) with high order elements such as Adini's elements for Poisson's equations with singularities. The best superconvergence rates $O(h^{3.5})$ in H^1 norm can be reached, where h is the maximal boundary length of elements. For the penalty and the penalty plus hybrid Trefftz methods, the bounds of effective condition numbers retain the same. However, for the penalty TM combination, not only are the algorithms simple, but also the numerical solutions are more accurate. Hence we prefer the penalty TM combination to the penalty plus hybrid TM combination. Such a recommendation is based on the effective condition number, but not on the traditional condition number, based on which the penalty TM combination was declined in Li and Yan [Global superconvergence for blending surfaces by boundary penalty plus hybrid FEMs for biharmonic equations. Appl Numer Math 2001;39:61–85]. The new stability analysis in this paper by means of effective condition number may re-evaluate the existing numerical methods; this paper reports the results of Motz's problem. Note that in Huang and Li [Effective condition number and superconvergence of Trefftz method coupled with high order FEM for singularity problems, Eng Anal Bound Elem 2006;30:270–83], no stability analysis was provided.

Keywords: Effective condition number; Condition number; Trefftz method; Coupling method; Global superconvergence; Singular problem; Motz's problem

1. Introduction

In this paper, we apply the effective condition number in Huang and Li [1] to the penalty Trefftz method (TM) with high order FEMs for singularity problems. Let the solution domain be a polygon, and S be divided by Γ_0 into two subdomains S_1 and S_2 without overlap, i.e., $S = S_1 \cup S_2 \cup \Gamma_0$ and $S_1 \cap S_2 = \emptyset$. Suppose that there exists a singularity on ∂S_2 and that the solution in S_1 is highly smooth. Let S_1 be divided again into small triangles $\triangle_{ij}$, i.e., $S_1 = \cup \triangle_{ij}$. We may choose the piecewise admissible functions,

$$v = \begin{cases} v^- = v_p & \text{in } S_1, \\ v^+ = \sum_{i=1}^{L} a_i \Psi_i & \text{in } S_2, \end{cases} \tag{1.1}$$

where v_p is the piecewise p-order polynomials of p-order elements, $\{\Psi_i\}$ are the known particular solutions, and a_i are the unknown coefficients. Some coupling techniques are needed to satisfy the interior continuity condition $u^+ = u^-$ on Γ_0. First, we consider the penalty techniques by

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using the following penalty integral,

$$\frac{P_c}{h^{2\sigma}}\int_{\Gamma_0}(v^+ - v^-)^2, \tag{1.2}$$

where the parameters $\sigma > 0$, $P_c > 0$, and h is the maximal boundary length of $\triangle_{ij}$. In Li [2, p. 229], the optimal convergence rates $O(h^p)$ in H^1 norm can only be achieved for $\sigma = 1$ and $p = 1$, i.e., the linear elements, unfortunately. To remedy this drawback, we invoke the penalty integral involving approximation,

$$\frac{P_c}{h^{2\sigma}}\widehat{\int}_{\Gamma_0}(v^+ - v^-)^2, \tag{1.3}$$

where the rule of quadrature is chosen as

$$\widehat{\int}_{\Gamma_0} uv = \int_{\Gamma_0}\hat{u}\hat{v}, \tag{1.4}$$

and $\hat{u}$ and $\hat{v}$ are just the p-order interpolant polynomials of u and v, respectively. Hence the optimal convergence rates $O(h^p)$ in H^1 can be achieved for p-order($p \geqslant 2$) elements if choosing $\sigma \geqslant p$. However, for the traditional condition number, the bounds as Cond $= O(h^{-2\sigma-2})$ are derived in this paper. Hence when $\sigma = p$, the Cond $= O(h^{-2p-2})$ is large, or even huge when $p \geqslant 3$. For Adini's elements, the superconvergence $O(h^{3.5})$ is obtained when $p = 3.5$, to give Cond $= O(h^{-9})$. It is due to the huge Cond that the penalty TM with high order FEMs was declined, see [3–5]. Instead, we invoke the penalty plus hybrid techniques

$$\frac{P_c}{h^{2\sigma}}\widehat{\int}_{\Gamma_0}(v^+ - v^-)^2 - \widehat{\int}_{\Gamma_0}\frac{\partial v^+}{\partial n}(v^+ - v^-). \tag{1.5}$$

The optimal convergence rates $O(h^p)$ by using (1.5) can be obtained by choosing $\sigma = 1$, to give the Cond $= O(h^{-4})$. For Adini's elements, the Cond $= O(h^{-9})$ for the penalty TM combination is deduced down to the Cond $= O(h^{-4})$ for the penalty plus hybrid TM combination. Hence, based on Cond, the penalty plus hybrid TM combination with high order FEMs is recommended. Such conclusions have also been drawn for the harmonic and the biharmonic equations to couple the complicated exterior boundary, see [3–5].

In fact, in many cases, the huge Cond is misleading, because it is too large to provide the true bounds of relative errors of perturbation. The better stability analysis may be made, based on the effective condition numbers Cond_eff given in [1]. Interestingly, the bounds of Cond_eff for both (1.3) and (1.5) are basically the same, to be Cond_eff $= O(h^{-1.5})$. For Motz's problem, Adini's elements are used in the subdomain where the solution is highly smooth, and the numerical solutions by the penalty TM combination are more accurate than those by the penalty plus hybrid TM. Note that in [1], no analysis for effective condition number was provided. Since the algorithms for (1.3) are also simpler than those for (1.5), the recommendation is switched to the penalty TM combination. Such a priority on the TM combinations is just contrary to that in our previous study, based on traditional Cond.

This paper is organized as follows. In Section 2, the new effective condition numbers in p-norm are defined. In Section 3, the penalty TM combinations with Adini's elements and p-order elements are discussed, and the error bounds are derived. In Section 4, the bounds of effective condition numbers as well as Cond are derived. In the last section, numerical experiments are carried out, to support the error and the stability analyses made.

2. Effective condition number

2.1. Definition

For solving the linear algebraic equations $\mathbf{Ax} = \mathbf{b}$, the traditional condition number is defined by Cond $= \sigma_1/\sigma_n$, where σ_1 and σ_n are the maximal and minimal singular values of matrix $\mathbf{A}$, respectively. The condition number is used to provide the bounds of relative errors from the perturbation of both $\mathbf{A}$ and $\mathbf{b}$. Such a Cond can only be reached by the worse situation of all rounding errors and all $\mathbf{b}$. For the given $\mathbf{b}$, however, the real relative errors may be smaller, and even much smaller than the Cond. In this case the effective condition number is called, and was studied in Chan and Foulser [6] and Christiansen and Hansen [7]. In fact, the effective condition number was first proposed for specific non-homogenous term in Rice [8] in 1981; but the natural condition number was called. In Huang and Li [1], we propose the new computational formulas for effective condition number Cond_eff, and define the new simplified effective condition number Cond_E. For the latter, we only need the eigenvector of the minimal eigenvalue for $\mathbf{A}$, which can be easily obtained by the inverse power method. This paper reports the important analysis of effective condition number for the combinations of Trefftz methods to solve Poisson's equation with singularities. In contrast, there was no stability analysis for the Trefftz combinations with FEM in [1].

Let us consider the linear algebraic equations

$$\mathbf{Ax} = \mathbf{b}, \tag{2.1}$$

where $\mathbf{A} \in R^{n\times n}$, $\mathbf{x} \in R^n$ and $\mathbf{b} \in R^n$ are the unknown and known vectors, respectively. When there occurs a perturbation of $\mathbf{b}$, the errors of $\mathbf{x}$ satisfy

$$\mathbf{A}(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{b} + \Delta\mathbf{b}. \tag{2.2}$$

The values of Cond are used to measure the relative errors of $\mathbf{x}$, which for (2.2) are given by

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} \leqslant \text{Cond} \times \frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}, \tag{2.3}$$

where $\|\mathbf{x}\|$ is the 2-norm and the matrix norm $\|\mathbf{A}\| = \sup_{\mathbf{x}\neq 0}\|\mathbf{Ax}\|/\|\mathbf{x}\|$. Let the matrix $\mathbf{A} \in R^{n\times n}$ be symmetric and positive definite. The eigenvalues are