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A symmetric boundary element model for the analysis of Kirchhoff plates

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ABSTRACT

The paper deals with the formulation and implementation of a new symmetric boundary element model for the analysis of Kirchhoff plates. The transversal displacement and normal slope boundary integral equations, usually adopted in the standard boundary element analysis, are considered together with bending moment, twisting moment and equivalent shear boundary integral equations. These equations are weighted by considering distributed sources related to the kinematic and static variables in the virtual-work sense. Moreover, particular attention is paid to the discretization of the boundary variables by shape functions selected in order to ensure continuity over the boundary and symmetry for the matrix system. The evaluation of the highly singular boundary integrals for overlapped integration domains is performed in closed form using a limit approach which provides self-contributions as limit values of non-singular terms. The corner effects and their treatment in the numerical procedure are also discussed. Various numerical examples for plates having different boundary conditions illustrate the performance of the model.

1. Introduction

Numerical models based on boundary discretization represent an efficient tool for the analysis of mechanical problems, providing compact numerical descriptions and accurate approximations of both the displacement and stress fields. Standard boundary models are usually based on boundary integral equations enforced at collocation points. The system entries are derived from the integration of the product between a shape function and a singular fundamental solution, usually associated with static sources. This approach leads to nonsymmetric systems.

Various collocation boundary element models have been developed to analyze the Kirchhoff plate problem. The first contributions came from Jaswon and Maiti [1] and Forbes and Robinson [2] for plates with smooth boundaries and Bézine [3] for plates including corner points. Further works were proposed by Altiero and Sikarskie [4], Stern [5] and by Hartmann and Zotemantel [6]. More recently contributions by Frangi [7], proposing a regularized boundary integral formulation, and by Aristodemo and Turco [8], dealing with the computational aspects involved in standard boundary element models, appeared.

The absence of symmetry in the collocation boundary element models hinders the analysis of dynamic problems and coupling between finite elements and boundary elements. Moreover, symmetric boundary formulations can take advantage of the availability of well-established numerical techniques developed in the context of finite element analysis.

Different methods can be adopted to generate a symmetric boundary element system. Some of them provide a symmetric system applying algebraic procedures to the collocation formulations while others employ suitable boundary formulations able to generate symmetric systems directly. The latter approach starts from integral equations associated with both static and kinematic distributed sources and generates the boundary system by the Galerkin method. Higher order singular kernels occurring in this case require the use of specialized methods for evaluating double integrals when integration domains overlap partially or totally. This computational aspect is more relevant in the boundary element analysis of Kirchhoff plates which involves hypersingular kernels.

The first paper on symmetric boundary element models for Kirchhoff plates came from Tottenham [9] who introduced a symmetric formulation obtained by weighting, in the Galerkin sense, the equations related to sources represented by forces, moments, bi-couples and tri-couples. Some years later a different symmetric boundary integral formulation, based on four equations corresponding to static and kinematic distributed sources, was presented by Hartmann et al. [10] for the analysis of plates without corner points. Successively, the contributions by Giroire and Nedelec [11] and by Nazaret [12], concerning plates with free edges, appeared. More recently, Frangi and Bonnet [13] developed a symmetric boundary element model for the analysis of polygonal plates with generic boundary conditions. The boundary

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equations are derived from the stationarity of an augmented potential energy functional, using Gauss transformations for regularizing the kernels and evaluating the boundary coefficients numerically. These transformations change the boundary variables typically used in the analysis of Kirchhoff plates.

In this paper a symmetric model for the boundary element analysis of Kirchhoff plates is presented. The paper gives a complete discussion of the formulation and the computational aspects involved in the construction of the numerical model. The aim of the paper is to provide an essential formulation, avoiding the manipulations associated with the regularization process and leaving the singularities to the analytical integration technique. In particular the boundary integral formulation considers five equations associated with two static sources (force and couple), two kinematic sources (normal slope and transversal displacement discontinuities) and a corner source (tangent rotation discontinuity). These equations are used in the Betti theorem in order to derive the boundary integral formulation which leads to symmetric systems in a straightforward way. The evaluation of the double integrals for overlapping domains is pursued by an entirely analytical integration method based on a limit approach. The coefficients having singular contributions are computed as limit values of non-singular integrals [14,19]. It is worth noting that this general approach was never applied to kernels having such orders of singularity. This process is developed on contiguous boundary elements by assuming interpolation functions which ensure inter-element continuity and vanish at the ends of the integration domains where the integration process locates the singularity poles.

After a brief review of the differential formulation of the Kirchhoff plate, the paper presents the integral formulation used for the symmetric boundary element analysis of polygonal plates. Successively, the integration process is sketched by referring to a typical boundary coefficient. Some tests for plates having different boundary conditions allow the numerical behavior of the proposed model to be evaluated in comparison with some refined collocation boundary element solutions.

2. Differential formulation of the Kirchhoff plate

A thin plate, defined over the domain Ω delimited by the boundary Γ, is considered, using a Cartesian reference system (O,x,y,z) located on its mid-surface (Fig. 1). The plate, having thickness h, is subjected to a transversal load $p[x,y]$.

In the Kirchhoff theory the assumption of negligible shear deformations leads to the description of the plate bending only in terms of the transversal displacement $w[x,y]$ [16,17]. The equilibrium over the domain can be expressed by the field equation

$$D\Delta\Delta w = p \tag{1}$$

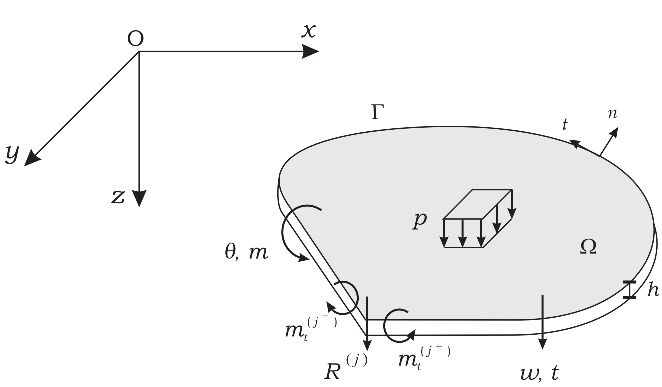

Fig. 1. Variables of the Kirchhoff plate model.

Δ being the Laplacian operator, E the elastic modulus, μ the Poisson coefficient and

$$D = Eh^3/12(1-\mu^2) \tag{2}$$

the flexural rigidity. At any regular boundary point the normal slope θ, the bending moment m, the shear q and the twisting moment m_t are defined by the expressions

$$\theta = w_{,n} \tag{3}$$

$$m = -D(w_{,nn} + \mu w_{,tt}) \tag{4}$$

$$q = -D(w_{,nnn} + \mu w_{,ntt}) \tag{5}$$

$$m_t = -D(1-\mu)w_{,nt} \tag{6}$$

where the comma denotes derivation with respect to the unit outward normal and t the unit tangent to the boundary Γ. The contraction of the static variables at smooth boundary points converts the shear q into the Kirchhoff shear t

$$t = q + m_{t,t} = -D(w_{,nnn} + (2-\mu)w_{,ntt}) \tag{7}$$

giving rise to the corner reactions R at the singular boundary point j

$$R^{(j)} = m_t^{(j^+)} - m_t^{(j^-)} \tag{8}$$

where the quantities $m_t^{(j^+)}$ and $m_t^{(j^-)}$ represent the twisting moments, furnished by Eq. (6), evaluated at the corner j on the forward and backward sides, respectively.

The differential problem above has to be solved integrating Eq. (1) on the basis of the boundary conditions

$$\begin{cases} w = \bar{w} & \text{or} \quad t = \bar{t} \\ \theta = \bar{\theta} & \text{or} \quad m = \bar{m} \end{cases} \tag{9}$$

where the barred symbols denote prescribed boundary values.

3. Boundary integral formulation

Boundary integral formulations of the Kirchhoff plate can be derived weighting the differential equation (1) by an appropriate test function w^* and integrating over the domain Ω [18]. By assuming the function w^* as the fundamental solution associated with sources (unit point actions) on the infinite plate and using the Gauss theorem, a description of the problem in terms of boundary variables alone is attained. Typical choices for these sources consist in a force F and a couple C, leading to the following displacement boundary integral equations:

$$cw + \int_\Gamma (t_F^* w + m_F^* \theta)\, d\Gamma + \sum_{j=1}^{n_c} R_F^{*(j)} w^{(j)} = \int_\Gamma (t w_F^* + m\theta_F^*)\, d\Gamma + \sum_{j=1}^{n_c} R^{(j)} w_F^{*(j)} + \int_\Omega p w_F^*\, d\Omega \tag{10}$$

$$c\theta + \int_\Gamma (t_C^* w + m_C^* \theta)\, d\Gamma + \sum_{j=1}^{n_c} R_C^{*(j)} w^{(j)} = \int_\Gamma (t w_C^* + m\theta_C^*)\, d\Gamma + \sum_{j=1}^{n_c} R^{(j)} w_C^{*(j)} + \int_\Omega p w_C^*\, d\Omega \tag{11}$$

where the starred quantities represent the fundamental solutions while w, θ, m, t and R denote the boundary fields. Factor c distinguishes sources located on the boundary ($c = \frac{1}{2}$) from sources inside the domain ($c = 1$) and n_c is the number of corners. The previous boundary integral equations are usually used to construct a standard collocation boundary element model [5,6].