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On the use of element-free Galerkin Method for problems involving incompressibility

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Abstract

The improvement of the performance of numerical methods, namely in the presence of incompressible deformations, have been, in the recent years, an important issue of discussion. However, incompressible or near incompressible behaviour has been a source of problems for numerical analysis of solid mechanics as the methods tend to present a locking response, or alternatively, spurious modes of deformation. The element-free Galerkin method (EFGM) is one of these methods, which exhibits some problems, namely, the so-called volumetric *locking* and *hourglass*. The first one is related with an inability to deform in some conditions. The hourglass is related with the apparition of non-physical eigenvectors. In this publication a new vision based in vector spaces theory is presented which can explain why these problems appear in the EFGM context.

Keywords: Element-free Galerkin method; Locking; Hourglass; Incompressibility

1. Introduction

Problems such as large deformations and/or dynamic fracture for which there are continuous changes in geometry implying an important remeshing effort have been studied by both methods: FEM and Meshless. The Meshless methods present some advantages which result mainly of the fact that refinements of the distribution of nodes are easier to perform than the remeshing procedure needed in the FEM.

In literature published in the last decade it is possible to find a large number of different developed Meshless methods: element-free Galerkin method (EFGM) [1, 2], where the approximants shape functions are constructed by means of a moving least-square procedure (MLS) first introduced by Nayroles et al. [3]. Reproducing Kernel particle methods [4], h-p Meshless Method [5] Multi-quadrics [6, 7], smooth particle hydrodinamics ([8] and Babuška et al. [9], proposed the method of the partitions of unity. Duarte and Oden [10] and Liu et al. [11] proved the convergence of the mesh-free methods. Zhu and Atluri [12] proposed a Petrov–Galerkin weak form in order to facilitate the computation of the integrals. However, there are some common problems involving both FEM and Meshless methods.

One of them is called *volumetric locking* and in the elasticity problems can appear when the coefficient of Poisson approaches the incompressibility limit 0.5. This situation appears, when the material is isochoric and the condition *div* $\mathbf{u}=0$ is locally imposed, being $\mathbf{u}$ the displacement field. *Shear locking* appears in dominant bending situations. In both situations, numerical problems arise, and the stiffness increases hindering the movement of nodes and so, the nodes remain fixed in their original positions in order to maintain the volume. To solve these situations, different mathematical tools have been developed. Mixed formulations split the material model into a deviatoric and a volumetric part, introducing therefore more variables. Reduced and selective-reduced integrations

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(SRI) schemes decrease the stiffness of the material, integrating the contribution of the hydrostatic pressure field of the stiffness matrix in certain points, fewer than the ones used for the non-hydrostatic part of the matrix. Malkus and Hughes [13] established the equivalence of both procedures under certain conditions for FEM. Later, the B-bar method was introduced by Hughes [14] and can be used as a mixed method or as a generalization of the SRI method. Vidal et al. [15] used a pseudo-divergence-free method combined with the EFGM. Askes et al. [16] established conditions for avoiding *locking* in the EFGM. González et al. [17], enriched the displacement basis in a natural neighbor Galerkin method. A method based on the concept of an enhanced strain field was introduced by Simo and Rifai [18]. For instance, César de Sá and Natal Jorge [19] proposed new enhanced strain elements within the scope of the analysis of the incompressible subspace of solutions. Later, Alves de Sousa et al. [20] extended this approach for 3D problems.

Another problem is related to the apparition of non-physical eigenvectors or so called *hourglass* forms [15]. To solve this problem several stabilization techniques have been developed in the FEM context. For instance, Reese and Wriggers [21] and Reese [22], split the element tangent matrix into constant and *hourglass* parts making a modal analysis on nodal level. The method of reduced integration plus *hourglass* stabilization has been derived to solve both *locking* and *hourglass*. In recent publications, *hourglass* stabilization is derived on the basis of a mixed method [23,24]. Chen et al. [25] developed a stabilized conforming nodal integration in the EFGM approximation.

An example of the utilization of vector spaces theory to explain the *locking* phenomena appears in César de Sá and Natal Jorge [19] in a FEM context. The aim of this work is to give a global explanation of why both problems arise in a formulation based in the EFGM. The vector spaces theory will be used to explain how the shape and dimension of a vector space can be used to interpret the occurrence of these phenomena in the EFGM context.

The present article is structured as follows. In Section 2, the EFGM is briefly described. In Section 3, the vector spaces theory applied to the interpretation of *locking* and *hourglass* problems in the EFGM context is proposed. In Section 4 simple numerical examples illustrate the theories referred previously, using different configurations with an integration cell and with the nodes over the borders of the integration cells used in the EFGM.

2. EFG method—a brief description

The EFGM employs moving least-square approximants to approximate the function $\mathbf{u}(\mathbf{x})$ with $\mathbf{u}^{\mathrm{h}}(\mathbf{x})$. These approximants are constructed by means of Eq. (1) where $\mathbf{p}(\mathbf{x})$ is a polynomial, $\mathbf{a}_j(\mathbf{x})$ are the unknown parameters and m is the order of the polynomial.

$$\mathbf{u}^{\mathrm{h}}(x_1, x_2, x_3) = \sum_{j=0}^{m} \mathbf{p}_j(\mathbf{x})\mathbf{a}_j(\mathbf{x}). \tag{1}$$

The unknown parameters $\mathbf{a}_j(\mathbf{x})$ are determined by means of the MLS method that minimizes the difference between the local approximation at any point and the nodal parameters $\mathbf{u}_I$:

$$\min\left(\sum_{I=1}^{N_{\text{nodes}}} w(\mathbf{x}-\mathbf{x}_I)[\mathbf{p}^{\mathrm{T}}(\mathbf{x}_I)\mathbf{a}(\mathbf{x}) - \mathbf{u}_I]^2\right), \tag{2}$$

where $w(\mathbf{x}-\mathbf{x}_I)$ is a weight function that determines the dominium of influence of a node. In this article, a cubic spline function is used as weight function and defined as:

$$w(r) = \begin{cases} \frac{2}{3} - 4r^2 + 4r^3 & \text{for} \quad r \leqslant \frac{1}{2}, \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3 & \text{for} \quad \frac{1}{2} < r < 1, \\ 0 & \text{for} \quad r > 1, \end{cases} \tag{3}$$

where r is the normalized radius:

$$r = \frac{||\mathbf{x}-\mathbf{x}_I||}{c_I \cdot d_{\max}} \tag{4}$$

with $d_{\max}$ being the scaling parameter, which is typically 2.0–4.0 for a static analysis [2], and c_I is the average distance between nodes.

Eqs. (1) and (2) according to [1] lead to:

$$\mathbf{u}^{\mathrm{h}}(\mathbf{x}) = \sum_{I=1}^{N_{\text{nodes}}} \phi_I(\mathbf{x})\mathbf{u}_I, \tag{5}$$

where the approximants are expressed by means of a linear combination of the nodal parameters, $\mathbf{u}_I$. In this case the multipliers are the shape functions of the EFGM. These functions do not satisfy the Kronecker delta criterion: $\phi_I(x_J) \neq \delta_{IJ}$ so they are approximants functions.

The aim is to know the field $\mathbf{u}^{\mathrm{h}}(\mathbf{x})$ which best approximates the real displacements field $\mathbf{u}(\mathbf{x})$. This last field verifies the equilibrium equation (or Cauchy equation) and the boundary conditions:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } \Omega, \tag{6}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{in } \partial_u \Omega, \tag{7}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{in } \partial_t \Omega, \tag{8}$$

where $\mathbf{n}$ is the unit normal vector, $\mathbf{b}$ is the body force, $\boldsymbol{\sigma}$ represents the Cauchy stresses and Ω is its domain which is bounded by $\partial_u \Omega$ and $\partial_t \Omega$.

To solve this system when a discretization of the continuum medium using a nodal distribution is done, a variational principle must be considered. There are different ways to reach this weak form: finding the critical point of a functional, by means of the principle of the virtual works, or using the residual of the equilibrium equation. An approach based in the virtual works is developed to establish the weak form of the equations.