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Research note

Axisymmetric multiquadrics

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Abstract

This paper reviews the previous axisymmetric global interpolation functions used in the context of the dual reciprocity boundary element method and dual reciprocity method of fundamental solutions connected to axisymmetric Laplace operator. It complements our axisymmetric thin plate splines [1] with the axisymmetric form of the Hardy's multiquadrics $(r^2+r_0^2)^{m/2}$; $m=\pm1$. This new functions can be used in the improved Golberg–Chen–Karur [2] type of approximations. The basic equations are accompanied by a set of related expressions that permit straightforward use of the developed global interpolation functions in a broad spectrum of dual reciprocity boundary element method and method of fundamental solutions, and meshless direct collocation like discrete approximate procedures.

Keywords: Boundary element method; Method of fundamental solutions; Collocation method; Radial basis functions; Dual reciprocity method; Multiquadrics; Axisymmetry

1. Introduction

Axisymmetric geometry and field problems occur very frequently in science and engineering. The discrete approximate solutions of the different governing equations in such situations are of pronounced importance. The fusion of the boundary element method and global interpolation emerges in a variety of dual reciprocity (DR) boundary element method (BEM) discrete approximative procedures [3,4] that give reasonable evaluations of the governing equations. Two very comprehensive overviews have been published [5,6] regarding the use of the different global approximation functions in the BEM context. However, the mathematical properties of such methods are nowadays far from being sufficiently understood. Because of the unresolved theoretical answers to related existence, uniqueness, convergence, and stability issues, many numerical experiments and comparisons have been traditionally made in an ad-hoc manner in the DRBEM literature.

The problem of global interpolation outside of the BEM context has been much more closely investigated mathematically [7]. Corresponding analyses show that the use of the radial basis class of functions [8] represents a proper choice for multidimensional global interpolation. Most of the related advances focus on the augmented thin plate splines (ATPS) and multiquadrics (MQ). The ATPS are known to give the minimized curvature of the interpolation and the MQ could, depending on the choice of the free parameter, converge very rapidly. Karur and Ramachadran [9] first gave DRBEM numerical examples with ATPS and claim a superior solution to the heuristic 'one-plus-r' global approximation functions in two-dimensional planar problems. Golberg et al. [2] used MQ in the method of fundamental solutions (DRMFS) variant with global interpolation. They demonstrate up to three orders of magnitude of improvement in accuracy over ATPS and 'one-plus-r' functions provided that the free parameter is properly chosen.

Surprisingly, not many DRBEM solutions structured with the fundamental solution of the Laplace equation deal with axisymmetric problems. In the pioneering work concerning this DRBEM aspect, Wrobel and Telles [10] heuristically use the global approximation functions of the form

$$_{\mathfrak{A}}\psi_n = \left((p_\rho - p_{n\rho})^2 + (p_z - p_{nz})^2\right)^{1/2}\left(1 - \frac{1}{4}\frac{p_\rho}{p_{n\rho}}\right), \tag{1}$$

with the notation elaborated in the next chapter. Masseé and Marcouiller [11] found this function inadequate and after

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several numerical experiments proposed

$$_{\mathfrak{A}}\psi_n = p_{n\rho}\left((p_\rho - p_{n\rho})^2 + (p_z - p_{nz})^2\right)\left(p_0\left((p_\rho - p_{n\rho})^2 + (p_z - p_{nz})^2\right)^{1/2} + p_\rho^3\right), \tag{2}$$

with p_0 representing a small positive constant which was set to 0.01.

The axisymmetric form of the scaled augmented thin plate splines has been developed in [1]. Its successful implementation and testing in the classical DRBEM is demonstrated in [12,13] where they appear in the context of solving the convective–diffusive problems with non-linear boundary conditions, material properties, and phase-change. These functions have been in addition used in DRBEM solving of the temperature field in DC casting of alluminium alloy billets [14] where they appear in a full-scale industrial context. Chen et al. [15] developed the solution of the Poisson equation based on the DRMFS. To make use of the MFS, it is necessary to calculate a particular solution, which can be subtracted off, so that the MFS can be used to solve the resulting Laplace problem. This presents a novel problem, since the axisymmetric Poisson operator does not have constant coefficients, so previous methods based on radial basis functions cannot be used. To overcome this, the source term is approximated by a two-dimensional polynomial in r and z as in Goldberg et al. [16]. One can then obtain polynomial particular solutions by the method of undetermined coefficients.

The principal incitements for this paper are two. The first is that the axisymmetric ATPS cannot be used in the context of transport phenomena that extend with one coordinate to infinity, because they do not decay with growing distance from the collocation point. Such arrangements are of extreme importance for example in environmental transport phenomena. The second fact is the fact that the axisymmetric form of MQ have not been deduced yet and can be applied instead of polynomials (for example in [15]).

The present paper thus focuses on a relatively complex derivation of the axisymmetic MQ and related expressions for use in the spectrum of DRBEM, DRMFS, and Kansa [17,18] like discrete approximate procedures.

2. Derivation

The interpolation of the scalar function $\mathcal{F} \in \Re^3$ with the three dimensional MQ $_3\psi_n$ could be represented [19,20] in the following form

$$\mathcal{F}(\mathbf{p}) \approx {}_3\psi_n(\mathbf{p})\zeta_n; \quad n = 1,2,\ldots,N+1, \tag{3}$$

where $\mathbf{p}$ stands for the position vector and N stands for the number of collocation points. The Einstein summation convention is used. The $N+1$ coefficients ζ_n are determined from the N collocation equations

$$\mathcal{F}(\mathbf{p}_i) = {}_3\psi_n(\mathbf{p}_i)\zeta_n; \quad i = 1,2,\ldots,N \tag{4}$$

and from the constraint

$$_3\psi_{N+1}(\mathbf{p}_i)\zeta_i = 0; \quad i = 1,2,\ldots,N. \tag{5}$$

The MQ of interest are

$$_3\psi_{n,1} = \sqrt{r_n^2 + r_0^2}, \tag{6}$$

$$_3\psi_{n,3} = (r_n^2 + r_0^2)^{3/2}, \tag{7}$$

$$_3\psi_{n,3} = \frac{1}{\sqrt{r_n^2 + r_0^2}}, \tag{8}$$

$$_3\psi_{n,-3} = \frac{1}{(r_n^2 + r_0^2)^{3/2}}, \tag{9}$$

with

$$r_n = |\mathbf{r}_n|, \mathbf{r}_n = \mathbf{p} - \mathbf{p}_n; \quad n = 1,2,\ldots,N, \tag{10}$$

where $\mathbf{P}_n$ stands for the position vector of collocation point n. The augmentation function is

$$_3\psi_{N+1} = 1. \tag{11}$$

The corresponding solutions of Poisson equation in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial_3\hat{\psi}_n}{\partial r}\right) = {}_3\psi_n \tag{12}$$

are, respectively

$$_3\hat{\psi}_{n,1} = \frac{1}{24}(2r_n^2 + 5r_0^2)\sqrt{r_n^2 + r_0^2} + \frac{r_0^4}{8r_n}\log R(r_n), \tag{13}$$

$$_3\hat{\psi}_{n,3} = \frac{1}{240}(8r_n^4 + 26r_n^2r_0^2 + 33r_0^2)\sqrt{r_n^2 + r_0^2} + \frac{r_0^6}{16r_n}\log R(r_n), \tag{14}$$

$$_3\hat{\psi}_{n,-1} = \frac{1}{2}\sqrt{r_n^2 + r_0^2} + \frac{r_0^2}{2r_n}\log R(r_n), \tag{15}$$

$$_3\hat{\psi}_{n,-3} = -\frac{1}{r_n}\log R(r_n), \tag{16}$$

where $R(r_n) \equiv r_n + \sqrt{r_n^2 + r_0^2}$. However, instead of working with above functions we prefer to introduce linear combinations $_3\psi_{n,A}$ and $_3\psi_{n,B}$ as follows

$$_3\psi_{n,A} = {}_3\psi_{n,-1} + \frac{r_0^2}{2}{}_3\psi_{n,-3} = \frac{1}{\sqrt{r_n^2 + r_0^2}} + \frac{r_0^2}{2}\frac{1}{(r_n^2 + r_0^2)^{3/2}}, \tag{17}$$

$$_3\psi_{n,B} = {}_3\psi_{n,1} - \frac{r_0^2}{4}{}_3\psi_{n,-1} = \sqrt{r_n^2 + r_0^2} - \frac{r_0^2}{4}\frac{1}{\sqrt{r_n^2 + r_0^2}}, \tag{18}$$

with corresponding solutions

$$_3\hat{\psi}_{n,A} = {}_3\hat{\psi}_{n,-1} + \frac{r_0^2}{2}{}_3\hat{\psi}_{n,-3} = \frac{1}{2}\sqrt{r_n^2 + r_0^2}, \tag{19}$$

$$_3\hat{\psi}_{n,B} = {}_3\hat{\psi}_{n,1} - \frac{r_0^2}{4}{}_3\hat{\psi}_{n,-1} = \frac{1}{12}(r_n^2 + r_0^2)^{3/2}. \tag{20}$$