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Crack analysis using an enriched MFS domain decomposition technique

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Abstract

In this paper we consider the application of the method of fundamental solutions to solve crack problems. These problems present difficulties, which are not only related to the intrinsic singular nature of the problem, instead they are mainly related to the impossibility in choosing appropriate point sources to write the solution as a whole. In this paper we present: (1) a domain decomposition technique that allows to express a piecewise approximation of the solution using a method of fundamental solutions applied to each subdomain; (2) an enriched approximation whereby singular functions (fully representing the singular behaviour around the cracks or other sources of boundary singularities) are used. An application of the proposed techniques to the torsion of cracked components is carried out.

Keywords: Method of fundamental solutions; Crack problems; Domain decomposition; Enriched approximations

1. Introduction

The Method of Fundamental Solutions (MFS) is a simple but powerful technique that has been used to obtain highly accurate numerical approximations of PDE solutions with simple codes and small computational effort. It has been introduced in the 1960 s (cf. [4,15]) as an alternative numerical method to the boundary integral methods in several elliptic homogeneous problems (e.g. [6,8,14,19], see also [2,9] for extensions to nonhomogeneous PDEs).

The application of the MFS can be justified by density results that have been established for simply connected domains with regular boundary (e.g. [2,6]); its generalization to multiply connected domains is straightforward. However, the application of the MFS, with external sources, cannot be justified if the domain includes cracks, that is, if boundary irregularities occur.

The study of the solution behavior in cracked domains is important in problems such as cracked bars under torsion [7], or to identify the presence of cracks inside a domain (e.g. [1,3]). The solution of crack problems presents difficulties due to its singular behavior at the crack. The MFS relies on the use of point-sources outside the domain (which are analytic functions inside the domain) whose linear combination cannot adequately model the singular behavior at the cracks.

- Here we present a model problem in potential theory where cracks are considered. One aim of this work is to investigate a possible application of this technique to the Cauchy–Navier equations of elasticity. In the current framework these cracks may also be seen as mode III cracks.
- Some previous works, e.g. [12,13] or [20], consider the application of the of method of fundamental solutions to singular problems. For instance, in [20] a mixed Dirichlet–Neumann boundary condition was considered, and in symmetric situations these mixed boundary conditions can be a simplified setting to solve crack problems.

On the other hand, the MFS can be seen as a Trefftz type method (cf. [22]), based on the use of a superposition of solutions of the PDE. The pure Trefftz method is based on an expansion of the solution as a series and it is also a simple computational method that has been applied to several PDE problems (e.g. [10]). More recently Trefftz methods have been applied to crack analysis (e.g. [17]), using a domain decomposition technique (c.f. [16]). The application of domain decomposition techniques together with meshless methods has been a subject of recent research in other contexts (e.g. [5,11,18]).

Here we consider a general domain decomposition technique applied to the method of fundamental solutions and

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a procedure to enrich the approximations. To the standard basis of functions used in the MFS (translations of the fundamental solution itself) a new set of singular functions is added so that problems exhibiting boundary singularities (such as crack problems) may be solved.

In particular, here we consider the problem of cracked bars under torsion as in [7]. We give several numerical tests that show high quality approximations of the singular solution, when using the enriched MFS, or even better when using an enriched MFS with a domain decomposition technique.

2. Statement of a general crack problem

Consider a crack (or several cracks) γ embedded in a bounded domain $\Omega \subset \mathbb{R}^d$ (we will be interested in dimensions $d=2$ or 3), with regular boundary $\Gamma=(\Omega$ In a general situation, we look for the solution u on Ω/γ of the boundary value problem

$$\begin{cases} \mathcal{D}u = 0 & \text{in } \Omega\backslash\bar{\gamma}, \\ \mathcal{B}_\gamma u = g & \text{on } \gamma, \\ \mathcal{B}_\Gamma u = f & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{1}$$

Here $\mathcal{D}$ stands for an elliptic differential operator, with fundamental solution Φ such that $\mathcal{D}\Phi=\delta$. The boundary condition on (Ω is given by the operator $\mathcal{B}_\Gamma$ and on the crack we consider a boundary condition given by the operator $\mathcal{B}_\gamma$. $\bar{\gamma}$ stands for the closure of γ, which includes its tips.

3. An enriched MFS for crack problems

The basic features of all Trefftz-based techniques [22] is the use, as approximating functions, of actual solutions of the homogeneous differential equation. In this context, the Method of Fundamental Solutions (MFS) may be seen as Trefftz technique. In both cases boundary-only collocation may be used, the difference being the location of the sources and the approximating functions used. In the MFS an arbitrary number of sources are located at an artificial boundary and the same function is used at all sources - *the fundamental solution*, whereas in the classical Trefftz collocation method an arbitrary number of functions, located at a single source, are used - *the so-called T-functions* (e.g. [10]).

The MFS is a simple technique that has been used to obtain highly accurate numerical approximations when the domain presents no cracks. The presence of cracks produces singular solutions near the crack tips that lead to difficulties in the approximation. The MFS relies on the use of point-sources outside the domain and the singular behavior near the crack tips cannot be reproduced by a combination of these analytic functions.

A technique to provide an approximation of the solution using the MFS is to add, to the regular approximation, singular particular solutions (e.g. [20]). This approach has also been used in the context of Trefftz methods (e.g. [17]) for crack problems.

Consider that the solution may be written as the sum of a regular and a singular solution as follows,

$$u(x) = u_\mathrm{R}(x) + u_\mathrm{S}(x), \quad \forall x \in \Omega.$$

This technique is based on an *a priori* knowledge of the behaviour of the singular solution itself; to appropriately model the singular behaviour, actual singular particular solutions are used.

–The approximation of u_R is made using regular solutions, i.e. the set of fundamental solutions $\{\phi_1,\ldots,\phi_{m_R}\}$ with $\phi_k(x)=\Phi(x-y_k)$ where the set of point-sources $Y=\{y_1,\ldots,y_{m_R}\}\subset\mathbb{R}^d\backslash\bar{\Omega}$ is usually considered at an artificial boundary $\hat{\Gamma}\subset\mathbb{R}^d\backslash\bar{\Omega}$ that encloses $\Gamma=(\Omega$

$$u_\mathrm{R}(x) \approx \sum_{k=1}^{m_\mathrm{R}} \alpha_k \phi_k(x).$$

These regular solutions verify $\mathcal{D}u_\mathrm{R}=0$ in the whole Ω

–On the other hand, the approximation of u_s is made using singular solutions, i.e. a set of solutions $\{\psi_1,\ldots,\psi_{m_S}\}$ that may be used to characterize the singular behaviour of the solution-near the crack,

$$u_\mathrm{S}(x) \approx \sum_{k=1}^{m_\mathrm{S}} \beta_k \psi_k(x). \tag{2}$$

These functions verify the domain equation, i.e. $\mathcal{D}u_\mathrm{S}=0$ in $\Omega\backslash\bar{\gamma}$.

3.1. Least-squares fitting

We consider two sets of collocation points $X_\Gamma=\{x_1,\ldots,x_{n_\Gamma}\}\subset\partial\Omega$ and $X_\gamma=\{\xi_1,\ldots,\xi_{n_\gamma}\}\subset\gamma$. The collocation system is

$$\begin{bmatrix} \mathcal{B}_\Gamma\Phi(x_i-y_j) & \mathcal{B}_\Gamma\Phi_j(x_i) \\ \mathcal{B}_\gamma\Phi(\xi_i-y_j) & \mathcal{B}_\gamma\Phi_j(\xi_i) \end{bmatrix} \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} = \begin{bmatrix} f(x_i) \\ g(\xi_i) \end{bmatrix}, \tag{3}$$

Imposing $n_\Gamma+n_\gamma>m_\mathrm{R}+m_\mathrm{S}$ we solve the system using discrete least squares, to obtain the approximation

$$\tilde{u}(x) = \sum_{k=1}^{m_\mathrm{R}} \alpha_k\phi_k(x) + \sum_{k=1}^{m_\mathrm{S}} \beta_k\psi_k(x)$$

that minimizes the error at the collocation points, i.e.:

$$E(\tilde{u}) = ||\mathcal{B}_\Gamma(\tilde{u})-f||_{l^2(X_\Gamma)} + ||\mathcal{B}_\gamma(\tilde{u})-g||_{l^2(X_\gamma)}.$$

4. Domain decomposition technique

To perform a domain decomposition technique that allows to express the solution u on γ with two layers, we divide the domain into two subdomains Ω_1 and Ω_2. These subdomains are such that $\bar{\Omega}_1\cup\bar{\Omega}_2=\bar{\Omega}$ and the interface $\Gamma_\gamma=\bar{\Omega}_1\cap\bar{\Omega}_2$ contains the crack γ. This interface Γ_γ is a $d-1$ dimensional manifold that separates the domain Ω in the two simply connected subdomains Ω_1,Ω_2. We will call *artificial interface*