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The finite cell method for tetrahedral meshes

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ABSTRACT

The recently proposed Finite Cell Method (FCM) is a combination of higher order Finite Element Methods (FEM) and the Fictitious Domain Concept (FDC). So far, the discretization of the structure under investigation has been based on hexahedral cells when applying the FCM. In the current paper, we extend the FCM to tetrahedral cells offering several advantages over the standard approach. If geometrically complex industrial problems have to be solved, often geometry-conforming tetrahedral meshes already exist. Thus, only micro-structural details that are important for the application, such as pores, need to be resolved by the FDC. Another significant advantage of tetrahedral cells over hexahedral ones is the capability for local mesh refinements. This property is of special interest for problems with sharp gradients and highly localized features where a fine mesh is inevitable. By means of the tetrahedral FCM we can easily analyze the influence of the relevant micro-structural details on the mechanical behavior. The geometry of the micro-structures can be obtained using computed tomography (CT) scans. The data from the CT-scans can then be included into the FCM model in a straightforward fashion.

In this paper, the performance and accuracy of the tetrahedral FCM is demonstrated using two examples. The first problem is rather academic and examines a cube with a spherical void. Here, we demonstrate that both the FCM and the FEM achieve the same rates of convergence. As a second example we consider a more practical problem where we investigate the influence of a pore on the stress distribution in an exhaust manifold of a diesel particulate filter (DPF). Again, we observe a very good agreement between the results computed using the FEM and the FCM, respectively.

1. Introduction

The Finite Element Method (FEM) is a versatile tool to analyze complex problems in various engineering disciplines. However, it suffers from the need for boundary-conforming discretizations. This constraint creates a severe bottleneck in the simulation process. In [1] it is stated that in complex industrial applications almost 80% of the overall analysis time is attributed to the transfer of a computer aided design (CAD) model into the discretized FE model. Therefore, only 20% of the time is dedicated to the analysis itself which agrees very well with the authors' experiences. From this predicament we can infer that a significant reduction of the overall analysis time that is needed to solve an engineering problem using FEM, could be achieved if the ratio between the real solution time and the time needed to execute the discretization process were decreased. One idea to overcome this problem is the application of the Isogeometric Analysis (IGA). Here, identical higher order spline basis functions are used for geometry description and approximation of the independent field variables [2]. A second idea worth mentioning is the so-called *analysis-aware* CAD modeling put forward by Cohen et al. [3]. Despite their respective advantages, we rely on a third idea to alleviate the mesh generation process.

From the authors' point of view, the most promising approach to solve this problem is based on applying the Fictitious Domain Concept (FDC) [4–7]. The fundamental idea of the FDC is to extend the physical domain beyond its possibly complex boundaries so that the generated embedding domain is larger than the original one but has a very simple geometry. Therefore, Cartesian grids can be used to discretize the domain in a structured manner. This procedure can be automated straightforwardly and thus the input required by the user is minimized. In this class of methods the Finite Cell Method (FCM) holds a special position [8,9] as it combines the advantages of higher order FEMs [10–15] with those provided by the FDC. Here, the numerical error is reduced by elevating the polynomial degree of the higher order shape functions (p-refinement). For smooth problems an exponential convergence can even be reached [10]. On the other hand, the mesh generation can be easily automated when employing the FDC [16,17]. These two advantages are inherited by the FCM [18,19].

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In the present contribution the idea of the FCM is extended from regular hexahedral grids to unstructured tetrahedral ones. In the remainder of the paper the proposed method is therefore referred to as the *tetrahedral FCM* or in short *tet-FCM*. In the context of industrial applications, a body-fitted discretization may already be available, hence the FDC is only applied for Regions of Interest (RoI) within the structure. That is to say, only micro-structural details are captured by means of the FDC. Although the advantage of a fully automated discretization is lost, we retain the possibility to investigate different micro-structures in parametric studies without remeshing the computational domain. To start, the idealized structure is investigated to provide the analyst with insight into the mechanical behavior which is used to define RoIs where the influence of discontinuities can be included by means of the FCM. A second motivation for the development of the tetrahedral FCM is the capability of tetrahedral elements to easily generate locally refined meshes. This property is of special interest for problems with sharp gradients and highly localized features where a fine mesh is required. To fully exploit the refinement capabilities of tetrahedra we are forced to employ unstructured grids, even though the extended domain is geometrically simple. However, such discretizations can easily be generated using existing mesh generators.

The paper is organized as follows: in Section 2 we provide a concise introduction to the fundamental principles of the conventional FCM. Then the basic ideas of the proposed tetrahedral FCM are discussed in Section 3. Here, we extend the FCM to geometry-conforming and -nonconforming discretizations in terms of tetrahedral finite cells. Moreover, the tetrahedral shape functions are introduced and the spacetree refinement scheme is discussed. In Section 4 we provide numerical examples that demonstrate the performance of this novel method. Finally, we summarize the key aspects of the present paper and briefly discuss ongoing research activities in Section 5.

2. The finite cell method

In the following, we briefly summarize the basic principles of the FCM. The point of departure for the derivation of the weak form are the mechanical equilibrium equations. Let us start considering a linear elasto-static problem, defined on the physical domain Ω, that is described by the following variational form

$$\mathcal{B}(\mathbf{u},\mathbf{v})=\mathcal{F}(\mathbf{v}),\quad \forall\ \mathbf{v}\in V, \tag{1}$$

in which V is the admissible space of all test functions. The bi-linear form $\mathcal{B}$ and the linear form $\mathcal{F}$ are defined as

$$\mathcal{B}(\mathbf{u},\mathbf{v})=\int_{\Omega}[\mathbf{L}\mathbf{v}]^{\mathrm{T}}\mathbf{C}[\mathbf{L}\mathbf{u}]\mathrm{d}\Omega, \tag{2}$$

$$\mathcal{F}(\mathbf{v})=\int_{\Omega}\mathbf{v}^{\mathrm{T}}\bar{\mathbf{f}},\mathrm{d}\Omega+\int_{\Gamma_N}\mathbf{v}^{\mathrm{T}}\bar{\mathbf{t}}\ \mathrm{d}\Gamma, \tag{3}$$

where $\mathbf{L}$ denotes the linear strain-displacement operator, $\mathbf{u}$ is the displacement vector, $\mathbf{v}$ represents the vector of unknowns of the test functions. Moreover, $\mathbf{C}$ stands for the elasticity matrix, $\mathbf{f}$ denotes the vector of body forces and $\mathbf{t}$ is the traction vector, whereas the bar over a variable $\bar{\cdot}$ signifies a prescribed value. The traction is defined on the Neumann boundary Γ_N of the structure. In addition to the weak form—Eq. (1)—we also have to apply boundary conditions as

$$\boldsymbol{\sigma}\mathbf{n}=\bar{\mathbf{t}}\quad \text{on } \Gamma_{\mathrm{N}}, \tag{4}$$

$$\mathbf{u}=\bar{\mathbf{u}}\quad \text{on } \Gamma_{\mathrm{D}}. \tag{5}$$

Here, $\boldsymbol{\sigma}$ denotes the stress tensor and $\mathbf{n}$ constitutes the outward normal vector of unit length. On the Dirichlet boundary Γ_D displacement boundary conditions are enforced.

As briefly mentioned in the Introduction— Section 1—a body-fitted discretization is required, if Eq. (1) is solved by means of the FEM. By exploiting the FDC [4–7], we circumvent this necessity. The main idea is that the physical domain under investigation is embedded into an extended domain Ω_{ex}, see Fig. 1. Generally speaking, Ω_{ex} is the union of the physical domain Ω with the fictitious domain Ω_{fic}. The main advantage of this approach is that the extended domain is of simple geometry and therefore, it can be discretized straightforwardly by means of regular Cartesian grids, see Fig. 2. This approach is, however, not limited to quadrilateral or hexahedral elements but can also be applied to triangular or tetrahedral elements as illustrated in Fig. 2. A detailed explanation of the algorithm is provided in Section 3. To distinguish between body-fitted finite elements and non-conforming ones, the term finite cell is used instead. In the sense of the FEM, the independent field variables are now approximated over the extended domain [8,9]. Accordingly, Eq. (1) is solved over Ω_{ex} and is given by

$$\mathcal{B}_{\mathrm{ex}}(\mathbf{u},\mathbf{v})=\mathcal{F}_{\mathrm{ex}}(\mathbf{v}),\quad \forall\ \mathbf{v}\in V. \tag{6}$$

In contrast to Eqs. (2) and (3), the bi-linear form $\mathcal{B}_{\mathrm{ex}}$ and the linear form $\mathcal{F}_{\mathrm{ex}}$ read

$$\mathcal{B}_{\mathrm{ex}}(\mathbf{u},\mathbf{v})=\int_{\Omega_{\mathrm{ex}}}[\mathbf{L}\mathbf{v}]^{\mathrm{T}}\alpha\mathbf{C}[\mathbf{L}\mathbf{u}]\ \mathrm{d}\Omega, \tag{7}$$

$$\mathcal{F}_{\mathrm{ex}}(\mathbf{v})=\int_{\Omega_{\mathrm{ex}}}\mathbf{v}^{\mathrm{T}}\alpha\bar{\mathbf{f}}\ \mathrm{d}\Omega+\int_{\Gamma_{\mathrm{N}}}\mathbf{v}^{\mathrm{T}}\bar{\mathbf{t}}\ \mathrm{d}\Gamma, \tag{8}$$

where α is the so-called indicator function that accounts for the geometry of the physical domain as

$$\alpha(\mathbf{x})=\begin{cases}1 & \forall\ \mathbf{x}\in\Omega\\ \alpha_0=10^{-q} & \forall\ \mathbf{x}\in\Omega_{\mathrm{ex}}\backslash\Omega.\end{cases} \tag{9}$$

Depending on the problem at hand, q is typically chosen in a range from 4 to 15. Theoretically, it is also possible to assign a value of 0 to α_0. However, this leads to ill-conditioning of the system matrices on a scale comparable to that of the behavior caused by choosing too large a penalty parameter to account for the Dirichlet boundary conditions. To avoid these conditioning problems, α_0 is set to a very small value that is close to zero. In this way, the variational formulation is stabilized and the energy contribution of the fictitious domain is weakly penalized [19]. According to [20], α_0 can be reliably determined with respect to the material properties

$$\alpha_0=(\lambda+\mu)\epsilon^*, \tag{10}$$

where λ and μ denote Lame's constants and ϵ^* is the standard unit roundoff.[1] Thus, a point in the fictitious domain is penalized by a small value of α. In [19] Dauge et al. provide a concise mathematical investigation of the FCM with respect to its convergence properties. In the mentioned paper it is also shown in which way the solution is influenced by the penalization parameter α_0. Moreover, they proved that exponential convergence can be achieved depending on the smoothness of the problem and the choice of α_0. If $\alpha_0=0$ we observe that Eqs. (1) and (6) are identical. We have to keep in mind however, that in contrast to the FEM, the mesh generation process is straightforward in the case of the FCM—see Fig. 2—and involves hardly any computational resources. On the other hand, it is clear that if we choose a too large value for α_0 the contribution of the fictitious domain to the weak form will not be negligible anymore and therefore the results will be arbitrarily inaccurate. In the FCM the effort needed to generate a geometry-conforming discretization is shifted to computing integrals with discontinuous integrands [9]. The computation

[1] ϵ^*: IEEE 754 machine precision $=2^{-53}\approx 1.16\cdot 10^{-16}$.