Finite Elements in Analysis and Design 113 (2016) 14–29

Contents lists available at ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.com/locate/finel

A fracture-controlled path-following technique for phase-field modeling of brittle fracture

N. Singh [a], C.V. Verhoosel [a,*], R. de Borst [b], E.H. van Brummelen [a]

[a] Department of Mechanical Engineering, Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands
[b] Department of Civil and Structural Engineering, University of Sheffield, Sheffield, UK

ARTICLE INFO

Article history:
Received 16 June 2015
Received in revised form
17 November 2015
Accepted 8 December 2015
Available online 22 January 2016

Keywords:
Brittle fracture
Phase-field modeling
Path-following methods
Staggered solution procedures

ABSTRACT

In the phase-field description of brittle fracture, the fracture-surface area can be expressed as a functional of the phase field (or damage field). In this work we study the applicability of this explicit expression as a (non-linear) path-following constraint to robustly track the equilibrium path in quasi-static fracture propagation simulations, which can include snap-back phenomena. Moreover, we derive a fracture-controlled staggered solution procedure by systematic decoupling of the path-following controlled elasticity and phase-field problems. The fracture-controlled monolithic and staggered solution procedures are studied for a series of numerical test cases. The numerical results demonstrate the robustness of the new approach, and provide insight in the advantages and disadvantages of the monolithic and staggered procedures.

1. Introduction

In many problems in brittle-fracture mechanics, phenomena such as nucleation, propagation, branching and merging occur. Complex crack patterns appear as a consequence of *e.g.* the presence of multiple cracks, anisotropy and heterogeneity. Using discrete fracture models it is generally difficult to capture such topologically complex crack patterns, which has led to the development of smeared or continuum crack models, including phase-field models [1,2]. In phase-field models the crack surface is regularized by a smeared damage (or phase-field) function, which avoids the need for the explicit tracking of fracture surfaces. Over the past years phase-field modeling of fracture has been applied to a wide range of problems, including dynamic fracturing [3,4], large deformation fracturing [5], fracturing of electromechanical materials [6], cohesive fracturing [7], and fluid-driven fracture propagation [8].

In this work we consider the quasi-static evolution of brittle fractures in an elastic solid, where fractures are driven by gradual incrementation of the loading conditions. Since softening and snap-back behavior are frequently encountered in such situations, path-following control is required to adequately track the complete equilibrium path [9]. Path-following techniques have been an indispensable tool in non-linear solid mechanics since the pioneering works of Riks [10], Crisfield [11] and Ramm [12]. While these path-following techniques were developed in the context of snap-back behavior caused by geometrical non-linearities, over the past decades various enhancements to the original path-following procedures have been proposed in order to increase their versatility and computational efficiency.

A particularly interesting application of path-following techniques is their use to track snap-back behavior as a result of material non-linearities, especially localized failure phenomena. In such situations the original path-following constraints have proven to lack robustness by the fact that they fail to account for the localized nature of the source of non-linearity. Various modified techniques have been proposed to account for this localized behavior, among which are a series of (semi-)automatic procedures for selecting degrees of freedom that contribute to the nonlinear behavior of the system [13,14]. Our work builds on the idea that an appropriate path-following technique can be obtained by selecting a physically motivated constraint equation. In this regard the crack mouth opening displacement (CMOD) and crack mouth sliding displacement (CMSD) control equations proposed by De Borst [15] can be considered as pioneering works. Inspired by these control equations energy-release rate path-following control was developed for the simulation of localized failure phenomena, including discrete cracking, smeared damage and softening plasticity [16,17]. The versatility of the energy-release rate control has been demonstrated for a variety of applications, including cases in which geometrical and material nonlinearities are competing [18].

When applied in the context of discrete fracture simulations, the energy release-rate path-following technique has the ability to indirectly control the rate at which a fracture propagates by proper selection of the energy dissipation increment. In Ref. [19] it has been shown that the energy-release rate control can be successfully applied to

* Corresponding author. Tel.: +31 40 247 2382.
E-mail addresses: n.singh@tue.nl (N. Singh), c.v.verhoosel@tue.nl (C.V. Verhoosel), r.deborst@sheffield.ac.uk (R. de Borst), e.h.v.brummelen@tue.nl (E.H. van Brummelen).

http://dx.doi.org/10.1016/j.finel.2015.12.005
0168-874X/

phase-field simulations, where the dissipation increment is related to the fracture-surface area increment through the critical energy release rate. In the case of phase-field simulations the relation between the path-following constraint and the fracture-surface area increase can be made explicit, *i.e.* the fracture-surface area can be expressed as a functional of the phase-field solution. This allows for direct prescription of the surface-area increments. This explicit dependence allows for the selection of the path-following parameter increment based on a criterion that relates the crack surface growth to the size of the employed (finite element) mesh, which provides a natural way of controlling the accuracy of the path-following scheme. In this work we formulate and study such a fracture-based path-following technique, which – if used in combination with a monolithic incremental-iterative path-following procedure – allows for the parametrization of the equilibrium path by specified fracture-surface area increments.

Quasi-static phase-field simulations of brittle fracture phenomena have mostly relied on the use of a staggered solution strategy, in which the elasticity problem and phase-field problems are decoupled [2]. This staggered solution strategy has been proven to be computationally efficient. A drawback of this solution strategy is that the step sizes need to be selected appropriately in order to control the accuracy of the procedure. The currently available staggered schemes are not capable of representing snap-back behavior. In this work a staggered fracture-based path-following method is derived from the monolithic scheme, which has the possibility of reducing the computational effort of the monolithic scheme at the cost of only satisfying the path-following increments in an approximate sense. This fracture-controlled staggered scheme does, however, inherit the property of the underlying monolithic scheme that the fracture propagation increments can directly be controlled (albeit in an approximate sense). This simplifies the selection of the step size compared to *e.g.* the displacement-based staggered scheme.

In Section 2 we introduce the phase-field formulation for brittle fracture and its discretization using the finite element method. In Section 3 we derive the fracture-based path-following constraint. In this section we also discuss various aspects of the corresponding incremental-iterative path-following procedure. In Section 4 we systematically derive the staggered path-following scheme, after which the monolithic scheme and staggered scheme are studied in detail in terms of computational effort and accuracy in Section 5. In this section we also study the nature of the snap-back behavior encountered in phase-field simulations for brittle fracture. Finally, conclusions are drawn in Section 6.

2. Phase-field formulation for brittle fracture

2.1. Problem formulation

We consider the evolution of a regularized fracture surface, $\Gamma_{l_c}(d)$, in an $n_{\text{dim}}-$dimensional elastic medium, $\Omega \subset \mathbb{R}^{n_{\text{dim}}}$, under quasi-static loading (see Fig. 1). The outward-pointing unit normal vector to the surface of the domain Ω is denoted by $\boldsymbol{n} : \Gamma \rightarrow \mathbb{R}^{n_{\text{dim}}}$. Small deformations and deformation gradients are assumed, and the deformation of the medium is described by the displacement field $\boldsymbol{u} : \Omega \rightarrow \mathbb{R}^{n_{\text{dim}}}$. The fracture surface, $\Gamma_{l_c}(d)$, is represented by the phase field $d : \Omega \rightarrow [0, 1]$, which approaches 1 inside a regularized crack and vanishes far away from the fracture surface. External tractions, $\bar{\boldsymbol{t}}$, are applied along the Neumann boundary Γ_N and prescribed displacements, $\bar{\boldsymbol{u}}$, are considered at the Dirichlet boundary Γ_D.

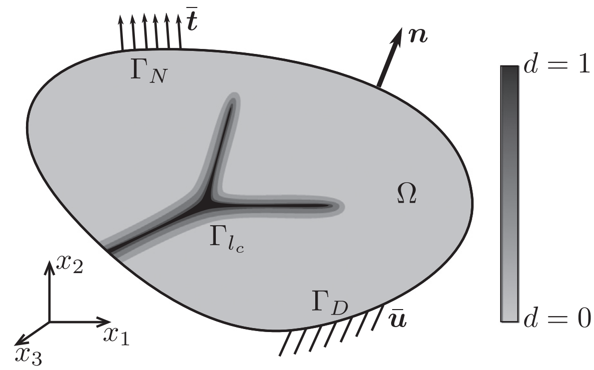

Fig. 1. Schematic representation of a domain Ω with regularized fracture surface $\Gamma_{l_c}(d)$ representing a fractured solid medium.

Under the above conditions, the strong form for the displacement and phase field is given by:

$$(S)\begin{cases} \nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0} & \text{in } \Omega \quad \text{(1a)} \\ \frac{\mathcal{G}_c}{l_c}\left(d - l_c^2 \Delta d\right) = 2(1-d)\mathcal{H} & \text{in } \Omega \quad \text{(1b)} \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}} & \text{on } \Gamma_N \quad \text{(1c)} \\ \boldsymbol{u} = \bar{\boldsymbol{u}} & \text{on } \Gamma_D \quad \text{(1d)} \\ \nabla d \cdot \boldsymbol{n} = 0 & \text{on } \Gamma \quad \text{(1e)} \end{cases} \tag{1}$$

In this strong form, $\mathcal{G}_c$ is the Griffith type critical energy-release rate and l_c is the length scale associated with the phase-field regularization of the fracture surface[1] (*i.e.* the width of the cracks) [1,2]. In order to restrict the fracturing process to tensile stress states, the Cauchy stress tensor in the above problem is defined as

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, d) = g(d)\boldsymbol{\sigma}_0^+(\boldsymbol{\varepsilon}) + \boldsymbol{\sigma}_0^-(\boldsymbol{\varepsilon}), \tag{2}$$

where $g(d) = (1-d)^2$ is the degradation function, $\boldsymbol{\sigma}_0^+$ and $\boldsymbol{\sigma}_0^-$ are the tensile and compressive parts of the virgin ($d=0$) Cauchy stress tensor [2], and $\boldsymbol{\varepsilon} = \nabla^s \boldsymbol{u}$ is the infinitesimal strain tensor. From the above stress definition it evidently follows that this degradation function must satisfy the conditions $g(0) = 1$ and $g(1) = 0$. The property that $g'(1) = 0$ ensures that the thermodynamic driving force for the phase-field model (*i.e.* the right-hand-side of the phase-field equation) vanishes once a fracture has completely evolved.

Irreversibility, *i.e.* the notion that the fracture surface can only extend ($\dot{\Gamma}_{l_c} \geq 0$), is enforced in the strong form (1) by means of the history field $\mathcal{H} : \Omega \rightarrow \mathbb{R}^+$. This history field satisfies the Kuhn–Tucker conditions for loading and unloading, defined as

$$\psi_0^+ - \mathcal{H} \leq 0, \quad \dot{\mathcal{H}} \geq 0, \quad \dot{\mathcal{H}}(\psi_0^+ - \mathcal{H}) = 0, \tag{3}$$

with ψ_0^+ the tensile part of the virgin elastic energy density.

2.2. Finite element discretization

To compute an approximate solution to the strong form (1) using the finite element method, the weak form is derived. Using the function spaces $\mathcal{V}^{\boldsymbol{u}} = \{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) | \boldsymbol{u} = \bar{\boldsymbol{u}} \text{ on } \Gamma_D\}$ and $\mathcal{V}^d = H^1(\Omega)$ for the trial functions, and $\mathcal{V}_0^{\boldsymbol{u}} = \{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) | \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_D\}$ and $\mathcal{V}^d$ for

[1] Here the length scale l_c is defined as in Ref. [2]. We note that in the literature sometimes use is made of the alternative length scale definition $\varepsilon = l_c/2$, *e.g.* [1].