Finite Elements in Analysis and Design 113 (2016) 43–53

Contents lists available at ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.com/locate/finel

Full Length Article

A composite mixed finite element model for the elasto-plastic analysis of 3D structural problems

Antonio Bilotta, Giovanni Garcea, Leonardo Leonetti *

Dipartimento di Ingegneria Informatica, Modellistica, Elettronica e Sistemistica, Università della Calabria, 87036 Rende (Cosenza), Italy

ARTICLE INFO

Article history:
Received 6 October 2015
Received in revised form
3 January 2016
Accepted 11 January 2016
Available online 1 February 2016

Keywords:
Mixed finite element
3D structural problems
Plastic collapse analysis
Elastoplasticity
Conic programming

ABSTRACT

The paper proposes a 3D mixed finite element and tests its performance in elasto-plastic and limit analysis problems. A composite tetrahedron mesh is assumed over the domain. Within each element the displacement field is described by a quadratic interpolation, while the stress field is represented by a piece-wise constant description by introducing a subdivision of the element into four tetrahedral regions. The assumptions for the unknown fields make the element computationally efficient and simple to implement also in existing codes. The limit and elasto-plastic analyses are formulated as a unified mathematical programming problem allowing the use of Interior Point like algorithms. A series of numerical experiments shows that the proposed finite element is locking free and has a good plastic behavior.

1. Introduction

The evaluation of the elastoplastic behavior of structures plays an important role in structural design. The finite element method is the most common numerical tool employed for the practical solution of many engineering problems including the evaluation of the plastic collapse state of structures. Improvements in the accuracy and reliability of finite element models for structural analysis have often been pursued using strategies based on complicated interpolations for the unknown fields. This can obscure the nature of the mechanical problem which is sometimes of simple interpretation. A more interesting approach may be the development of essential models [1,2] capable of capturing the main features of the structural behavior which could work well in a wide context. The plastic collapse state of a structure is, for example, characterized by a continuous displacement field and by strong concentrations and discontinuities in the plastic strain field. By simply considering this basic aspect a successful finite element (FE) model could be built [3]. The robustness of the numerical process, the accuracy with respect to all the mechanical fields involved in the numerical formulation and the applicability to a wide range of geometrical and numerical data are other desirable features of the model.

Another important aspect of the numerical model is its capability in analyzing structural problems with constraints on the possible strains, such as near incompressibility which concerns the elastic behavior of rubber-like materials and the broad field of elastic–plastic analysis when the plastic behavior is controlled by the deviatory part of the stress tensor and in bending problems. Compatible finite elements are usually unable to ensure the features described above and often they suffer from locking phenomena related to the choice of interpolations which are inadequate to describe internally constrained strains [4,5].

Several formulations have been proposed as alternatives to the compatible one in the context of linear elasticity. These include: the assumed stress formulations that, starting from the pioneering work by Pian [6], have been developed by several authors [7–9]; the mixed-enhanced elements that use additional strain fields to augment the compatible strain fields from the simplex interpolations [10]; elements based on the average nodal pressures/strains that compute the average volumetric strains or strains at nodes based on surrounding triangles or tetrahedrals [11,12]; the average nodal strain approach that alleviates the locking due to bending [13,14]. An assumed-strain finite element technique for the solution of plasticity problems stemming from the Nodally Integrated Continuum Element (NICE) formulation is presented in [12]. Another important family of FEM models is represented by the so-called smoothed-FEM [15] in which the average strain is assumed on the conflict faces of a background tetrahedral grid. In this category we also find the Edge-Smoothed and Node-Smoothed finite element model, developed for 2D problems, which assumes an average strain in a conflicting region associated to each edge or

* Corresponding author.
E-mail addresses: antonio.bilotta@unical.it (A. Bilotta), giovanni.garcea@unical.it (G. Garcea), leonardo.leonetti@unical.it (L. Leonetti).
URL: http://www.unical.it (L. Leonetti).

http://dx.doi.org/10.1016/j.finel.2016.01.002
0168-874X/

node of a background triangular mesh [16,17]. Smoothed elements have also been used successfully in plasticity problems [18,19].

In the analysis of elastic–plastic problems the finite element model should ensure even greater accuracy in determining the stress field and should be able to truly represent the strong strain concentration in the presence of the incremental collapse mechanism and then discontinuous fields. The presence of discontinuities makes many standard formulations, which furnish results sometimes poor in localizing the plastic regions and sensitive to the mesh pattern, of little use. In particular compatible elements with a point-wise enforcement of the plastic admissibility conditions in a series of Gauss points, even if still the most popular approach [20] are, in this respect, not the best possible choice. Recently, several mixed or generalized mixed formulations have been developed for 2D elasto-plastic problems. By formulating the optimality conditions of the elastoplastic holonomic step in a weak form, a class of three field elements has been proposed in [3,21] that allow discontinuities of the plastic multipliers and then of the plastic strains, within the element.

In this work a mixed finite element model which uses the assumption of an average stress field in the fashion of smoothed elements, also following the idea underlying the construction of composite elements [22,1], is proposed. It has been designed with the aim of describing solutions which can include the effects of discontinuous plastic strains as in [3]. The element is very easy to formulate and implement because it is based on simple assumptions for the displacement and stress fields, without any burden deriving from the management of a background mesh. As the internal variables are locally defined to the element, the proposed FEM model gives discrete operators with a smaller bandwidth with respect to existing smoothed FEMs. Furthermore it maintains the same algebra as a standard mixed element which makes it simple to implement also in existing codes. As the numerical results show, the element achieves the favorite framework in the plastic range where it takes advantage of the absence of volumetric locking, the capability of representing discontinuities within the element and the rather fine meshes required to describe complex data and constitutive discontinuities. Another great advantage is the computational efficiency which derives from the block-like shape of the complementary operator for which the inversion is trivial with respect to other FEM models for computing the algorithmic tangent operator.

The paper is organized as follows. Section 2 introduces the notation of the elastoplastic problem. Section 3 presents the assumed interpolations for the displacement and stress fields, reporting the closed-form expression of the discrete compatibility/equilibrium operator. Section 4 presents a unified variational formulation for both the limit analysis and the discrete holonomic elastoplastic step of a standard strain-driven formulation. The numerical results relative to a series of benchmarks are presented in Section 5. Finally Section 6 reports the conclusions.

2. Basic rules of the elastoplastic theory

We denote with $\boldsymbol{\sigma}[\boldsymbol{x}]$ and $\boldsymbol{\varepsilon}[\boldsymbol{x}]$ the stress and strain fields, and with $\boldsymbol{x}$ a point of the body domain Ω. Assuming perfectly plastic material, and omitting the dependence on $\boldsymbol{x}$ to simplify the notation, the stress $\boldsymbol{\sigma}$ is restrained to belong to the fixed admissible *Elastic Domain*

$$\mathbb{E} \equiv \{\boldsymbol{\sigma} : f[\boldsymbol{\sigma}] \le 0, \text{with } f[\mathbf{0}] < 0\} \tag{1}$$

where $f[\boldsymbol{\sigma}]$ is a convex yield function. $\mathbb{E}$ will be closed and convex and its boundary $\partial\mathbb{E}$ is characterized by $f[\boldsymbol{\sigma}] = 0$.

The strain increment $\dot{\boldsymbol{\varepsilon}}$ can be subdivided into the elastic part $\dot{\boldsymbol{\varepsilon}}^e$ and the plastic part $\dot{\boldsymbol{\varepsilon}}^p$. The elastic part is linearly linked to the stress through the *elastic law*:

$$\dot{\boldsymbol{\sigma}} = \mathbf{F}^{-1}\dot{\boldsymbol{\varepsilon}}^e = \mathbf{F}^{-1}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p) \tag{2}$$

where $\mathbf{F}$ is the compliance operator, symmetric and positive definite. $\dot{\boldsymbol{\varepsilon}}^p$ can be other than zero only if the stress $\boldsymbol{\sigma} \in \partial\mathbb{E}$ and is defined by the *plastic flow rule*:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\boldsymbol{n}[\boldsymbol{\sigma}], \quad \dot{\gamma} \ge 0, \quad \dot{\gamma}\dot{f} = 0, \quad \dot{\gamma} f = 0 \tag{3}$$

vector $\boldsymbol{n}[\boldsymbol{\sigma}]$ being contained in the sub-differential $\partial f[\boldsymbol{\sigma}]$ of f in $\boldsymbol{\sigma}$, for regular yield functions we have

$$\boldsymbol{n}[\boldsymbol{\sigma}] = \frac{\partial f[\boldsymbol{\sigma}]}{\partial \boldsymbol{\sigma}} \tag{4}$$

In these hypotheses, the following Drucker condition holds:

$$(\boldsymbol{\sigma}_y - \boldsymbol{\sigma})^T \dot{\boldsymbol{\varepsilon}}^p \ge 0 \quad \forall \boldsymbol{\sigma} \in \mathbb{E} \tag{5}$$

$\boldsymbol{\sigma}_y$ and $\dot{\boldsymbol{\varepsilon}}^p$ being related by the flow rule (3), (4).

2.1. The yield functions

Two standard yield functions will be considered. The classical H.V. Mises yield functions expressed as [20]

$$f[\boldsymbol{\sigma}] \equiv J_2[\boldsymbol{\sigma}] - \tau_0^2 \le 0 \quad \text{with } \tau_0^2 = \frac{\sigma_0^2}{3} \tag{6}$$

where σ_0 is the uniaxial yield stress, $J_2[\boldsymbol{\sigma}]$ is the second invariant of the deviatoric stress tensor, $3J_2[\boldsymbol{\sigma}] = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma}$ and

$$\boldsymbol{P} = \begin{bmatrix} 2 & -1 & -1 & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ -1 & -1 & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 6 & \cdot & \\ \cdot & \cdot & \cdot & \cdot & 6 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 6 \end{bmatrix}, \tag{7}$$

To study *pressure-sensitive materials* [20] we also introduce the Drucker–Prager conical yield surface in which the effect of the first invariant $I_1[\boldsymbol{\sigma}]$ of the stress tensor is considered

$$f[\boldsymbol{\sigma}] \equiv \alpha I_1[\boldsymbol{\sigma}] + \sqrt{J_2[\boldsymbol{\sigma}]} - \tau_0 \le 0 \tag{8}$$

where α and τ_0 are material parameters obtained from the friction angle φ and the cohesion c as

$$\alpha = \frac{3 \tan \varphi}{\sqrt{9 + 12 \tan \varphi^2}}, \quad \tau_0 = \frac{3c}{\sqrt{9 + 12 \tan \varphi^2}},$$

3. Interpolations of the displacement and stress fields

With reference to 3D problems, a composite tetrahedral mesh is assumed over the domain lying in the (x, y, z) space. Within each tetrahedral element the displacement field is described by a continuous interpolation expressed in terms of parameters located on its boundary sides, while the stress field is represented by a discontinuous description obtained by introducing a subdivision of the element into four tetrahedral regions. Each face of the element is a triangle with six nodes [23].

3.1. The mixed 3D finite element

The tetrahedral volume co-ordinate system makes the description of the relevant quantities of the discrete model compact and simple. The tetrahedral co-ordinates $(\xi_1, \xi_2, \xi_3, \xi_4)$ are related to the Cartesian ones (x, y, z) by the transformation [23]