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Elastoplastic implicit integration algorithm applicable to both plane stress and three-dimensional stress states

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ABSTRACT

An elastoplastic implicit integration algorithm applicable to both plane stress and three-dimensional stress states is developed for a general class of combined nonlinear kinematic–isotropic hardening models. The algorithm is first built for three-dimensional stress states in a general manner using the return mapping procedure and the Newton–Raphson method. The plane stress constraint is then incorporated into the Newton–Raphson iteration loop derived for three-dimensional stress states. The resulting algorithm has a mode patch that makes the algorithm applicable to both plane stress and three-dimensional stress states. The algorithm is specified by assuming an advanced evolution model of multiple back stresses, and is verified by performing numerical tests using plane stress, shell, and brick elements. The numerical tests are finite element analyses of homogeneously deformed plates and a cyclically loaded single-hole plate. It is demonstrated that the developed algorithm provides the quadratic convergence of iterations for implicit stress integration in plane stress, shell, and brick elements. It is also demonstrated that the algorithm is stable even in large incremental steps.

1. Introduction

A well-known model in the study of cyclic plasticity is the nonlinear kinematic hardening model proposed by Armstrong and Frederick [1]. This model has been highly rated because of its simple structure, consisting of strain hardening and dynamic recovery, and its superior ability to represent well the shapes of stress–strain hysteresis loops under cyclic loading [2]. Implementation of the model in finite element methods has been investigated in several studies [3–8]. The Armstrong and Frederick model has thus been available as a constitutive model of cyclic plasticity in commercial software of finite element analysis.

The Armstrong and Frederick model, however, usually overpredicts ratcheting and cyclic stress relaxation, which are fundamental phenomena in cyclic plasticity. In the last two decades, many studies have shown that this drawback can be more or less overcome by elaborating the dynamic recovery of back stress, as reviewed in Refs. [9–11]. For example, Ohno and Wang [12,13] considered step- and power-function nonlinearities in the dynamic recovery of back stress to improve the simulation of ratcheting and cyclic stress relaxation.[1] Jiang and Sehitoglu [16] further studied the power-function nonlinearity to simulate ratcheting. Investigation of these nonlinearities has led to successful simulations of ratcheting experiments [17–24] and easy identification of the material parameters of nonlinear kinematic hardening [18,25]. The consideration of such nonlinearities has also triggered computational studies that have implemented a general class of nonlinear kinematic hardening models in finite element methods [26–30], although only three-dimensional stress states have been dealt with in these studies.

The implicit integration of stress and the algorithmic expression of tangent stiffness are necessary to implement a constitutive model in the implicit method of elastoplastic finite element analysis [31–33]. For plane stress states, these need to satisfy the condition that the out-of-plane components of stress are zero. Because of this condition, which is called the plane stress constraint, particular schemes have been developed for plane stress elastoplastic finite element analysis [33]. An elegant scheme is that based on plane stress-projected constitutive models, which include only in-plane stress and strain components to satisfy the plane stress constraint [32,33]. This scheme

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[1] Chaboche [14] introduced a threshold in the dynamic recovery of back stress to improve the simulation of racheting. Henshall and Miller [15] considered power-function nonlinearity in the dynamic recovery of short range back stress.

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was originally proposed for the von Mises isotropic hardening model by Simo and Taylor [34] and Jetteur [35]. For complex constitutive models, however, it may not be easy or feasible to derive plane stress-projected models, and in such cases, it is recommended to use other schemes [33].

In this study, considering a general class of combined nonlinear kinematic–isotropic hardening models of plasticity, we develop an elastoplastic implicit integration algorithm that is applicable to both plane stress and three-dimensional stress states. To this end, the algorithm is first built for three-dimensional states in a general manner on the basis of the return mapping procedure and the Newton–Raphson method. The plane stress constraint is then incorporated into the Newton–Raphson iteration loop derived for three-dimensional stress states. The resulting algorithm has a mode patch that makes the algorithm applicable to both plane stress and three-dimensional stress states. To verify the developed algorithm, assuming an advanced evolution model of multiple back stresses, we perform numerical tests using plane stress, shell, and brick elements. The numerical tests are finite element analyses of homogeneously deformed plates and a cyclically loaded single-hole plate.

Throughout this paper, $\mathbb{I}$ and $\mathbb{O}$ indicate the fourth-rank unit and null tensors, while $\mathbf{1}$ and $\mathbf{0}$ signify the second-rank unit and null tensors; $\mathbb{I}^D$ denotes the deviatoric operator defined as $\mathbb{I}^D = \mathbb{I} - (1/3)\mathbf{1}\otimes\mathbf{1}$. Moreover, a superposed dot indicates differentiation with respect to time, dots stand for inner products between tensors (e.g., $\boldsymbol{\sigma}\mathbf{:}\boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ij}$ and $\mathbb{D}\mathbf{:}\boldsymbol{\varepsilon} = D_{ijkl}\varepsilon_{kl}$), and $\| \ \|$ denotes the Euclidean norm of second-rank tensors (e.g., $\|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma}\mathbf{:}\boldsymbol{\sigma})^{1/2}$).

2. Constitutive relations

In this study, we consider elastoplastic materials that are rate independent and initially isotropic. We presume that the strain $\boldsymbol{\varepsilon}$ is small and is additively decomposed into the elastic part $\boldsymbol{\varepsilon}^e$ obeying isotropic Hooke's law and the plastic part $\boldsymbol{\varepsilon}^p$ governed by the associated flow rule based on a yield surface $F=0$, which translates and expands. For simplicity, we consider isothermal conditions. We then have

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p, \tag{1}$$

$$\boldsymbol{\sigma} = \mathbb{D}^e : \boldsymbol{\varepsilon}^e, \tag{2}$$

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial F}{\partial \boldsymbol{\sigma}}, \tag{3}$$

$$F = (3/2)\|\mathbf{s} - \mathbf{a}\|^2 - Y^2, \tag{4}$$

where $\boldsymbol{\sigma}$ denotes the stress tensor, $\mathbb{D}^e$ the isotropic elastic stiffness tensor, $\dot{\lambda}$ the scalar to be determined using the consistency condition $\dot{F}=0$, $\mathbf{s}$ the deviatoric part of $\boldsymbol{\sigma}$, $\mathbf{a}$ the center of the yield surface in the deviatoric space, and Y the radius of the yield surface.

We assume that the yield surface $F=0$ expands as a function of accumulated plastic strain p:

$$Y = Y(p), \tag{5}$$

where

$$\dot{p} = \sqrt{2/3}\|\dot{\boldsymbol{\varepsilon}}^p\| \tag{6}$$

The center $\mathbf{a}$ can be regarded as the deviatoric part of back stress $\boldsymbol{\alpha}$. It is then appropriate to decompose $\mathbf{a}$ into several parts [2]:

$$\mathbf{a} = \sum_{i=1}^{M} \mathbf{a}^{(i)}, \tag{7}$$

where M denotes the number of multiple back stresses. We further assume that each $\mathbf{a}^{(i)}$ evolves as

$$\dot{\mathbf{a}}^{(i)} = \mathbf{f}^{(i)}(\mathbf{a}^{(i)}, p, \dot{\boldsymbol{\varepsilon}}^p), \tag{8}$$

where $\mathbf{f}^{(i)}$ is a material function that satisfies

$$\mathbf{f}^{(i)}(\mathbf{a}^{(i)}, p, \mathbf{0}) = \mathbf{0} \tag{9}$$

The multiple back stresses based on Eqs. (7) and (8) can be interpreted in terms of the nested multiple loading surfaces in deviatoric stress space [36].

3. Backward Euler discretization

Let us consider the loading interval between two states n and $n+1$, in which the constitutive variables have their values indicated by subscripts n and $n+1$. Let us use a prefix Δ to signify the changes in constitutive variables in the loading interval. The backward Euler method then allows Eqs. (1)–(8) to be discretized as

$$\boldsymbol{\sigma}_{n+1} = \mathbb{D}^e : \left(\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}^p_{n+1}\right), \tag{10}$$

$$\boldsymbol{\varepsilon}_{n+1} = \boldsymbol{\varepsilon}_n + \Delta\boldsymbol{\varepsilon}_{n+1}, \tag{11}$$

$$\boldsymbol{\varepsilon}^p_{n+1} = \boldsymbol{\varepsilon}^p_n + \Delta\boldsymbol{\varepsilon}^p_{n+1}, \tag{12}$$

$$\frac{\Delta\boldsymbol{\varepsilon}^p_{n+1}}{\Delta p_{n+1}} = \frac{3}{2}\frac{\mathbf{y}_{n+1}}{Y_{n+1}}, \tag{13}$$

$$\Delta p_{n+1} = \sqrt{2/3}\|\Delta\boldsymbol{\varepsilon}^p_{n+1}\|, \tag{14}$$

$$\mathbf{y}_{n+1} = \mathbf{s}_{n+1} - \mathbf{a}_{n+1}, \tag{15}$$

$$\mathbf{a}_{n+1} = \sum_{i=1}^{M} \mathbf{a}^{(i)}_{n+1}, \tag{16}$$

$$\Delta\mathbf{a}^{(i)}_{n+1} = \mathbf{f}^{(i)}\left(\mathbf{a}^{(i)}_{n+1}, p_{n+1}, \Delta\boldsymbol{\varepsilon}^p_{n+1}\right), \tag{17}$$

where $Y_{n+1} = Y(p_{n+1})$ and $p_{n+1} = p_n + \Delta p_{n+1}$. It is noted that Eqs. (10)–(17) apply only to elastoplastic loading intervals in which the following yield condition is satisfied:

$$F_{n+1} = (3/2)\|\mathbf{y}_{n+1}\|^2 - Y^2_{n+1} = 0 \tag{18}$$

It is also noted that Eq. (13) is based on the coaxiality of $\Delta\boldsymbol{\varepsilon}^p_{n+1}$ and $\mathbf{y}_{n+1}$, a consequence of Eqs. (3)–(5), as well as on Eqs. (14) and (18).

4. Implicit stress integration

This section describes an implicit stress integration algorithm valid for the constitutive relations given in Section 2. The algorithm is first built in a general manner in three-dimensional states. The plane stress constraint is then incorporated in the algorithm so that the algorithm is also applicable to plane stress states.

4.1. Three-dimensional stress state

The problem considered here is stated as follows. Given $\Delta\boldsymbol{\varepsilon}_{n+1}$ in addition to all constitutive variables for state n, find $\boldsymbol{\sigma}_{n+1}$ that satisfies the discretized constitutive relations (10)–(18). We use the return mapping procedure, which consists of an elastic predictor and a plastic corrector [31–33].