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Comparison of dual-mixed *h*- and *p*-version finite element models for axisymmetric problems of cylindrical shells

Balázs Tóth*, Lajos G. Kocsán

Department of Mechanics, University of Miskolc, H-3515, Miskolc-Egyetemváros, Hungary

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ABSTRACT

Dimensionally reduced cylindrical shell models using complementary energy-based variational formulations of *a priori* non-symmetric stresses are compared. One of them is based on the three-field dual-mixed Hellinger–Reissner variational principle, the fundamental variables of which are the stress tensor, the rotation and displacement vectors. The other one is derived from the two-field dual-mixed Fraeijs de Veubeke variational principle in terms of the self-equilibrated stress field and rotations. The most characteristic properties of the shell models are that the kinematical hypotheses used in the classical shell theories are not applied and the unmodified three-dimensional constitutive equations are employed. Our investigations are restricted to the axisymmetric case. The developed dual-mixed *hp* finite element models with C^0 continuous tractions and with discontinuous rotations and displacements are presented for bending–shearing (including tension–compression) problems. The computational performance of the constructed shell elements is compared through two representative model problems. It is numerically proven that no significant differences can be experienced between the two well-performing shell elements in the convergence rates.

1. Introduction

The widely used standard displacement finite element technique has some significant disadvantages. Firstly, the conventional finite element models provide worse convergence rates and lower accuracy in the evaluation of the stress field, which is often more important in many engineering applications than the knowledge of displacements. Secondly, the primal-mixed and displacement-based methods can lead to especially poor numerical results, when the plate and shell problems become bending dominated for small thicknesses. This phenomenon is known as numerical lockings including the shear- and membrane locking [1–5]. Thirdly, for more general models, arising in viscoelasticity and plasticity, the direct approximation of the stress field may be needed [6]. Furthermore, stability problems can be expected at nearly incompressible material, when the Poisson ratio is close to the incompressibility limit of 0.5. This is the reason why the material stiffness tensor becomes singular, i.e., the inverse of the stiffness matrix does not exist. The displacements can be inaccurately computed and even worse values can be obtained for the sum of the normal stresses. This is the well-known incompressibility locking effect [7,8].

There are several attempts to circumvent locking effects in the standard low-order displacement-based finite element models. These *h*-type shell elements are usually based on the various modifications of the principle of virtual work [9–13].

One of the most effective strategies for avoiding locking problems in the displacement-based finite element formulation is to use high-order (*p*-version) methods and the hierarchical plate and shell elements [8,14–17]. These *p* elements are verified to be locking-free in the energy norm and displacement computations for general shells, but the numerical results obtained for stresses are not exempt from locking, see for example [18,19]. It is important to note here that the *h*-version of the higher-order finite element schemes does not always converge asymptotically [20].

Another alternative way is to apply dual-mixed variational principles in the framework of the linear elasticity. These methods are suitable for the direct approximation of the stress field. Complementary energy-based dual-mixed variational formulation and finite element models have several advantages, as pointed out by [21,22]. They can provide better convergence rates and higher accuracy for the stresses than strain energy-based primal-mixed formulations and the conventional displacement-based formulations.

The classical dual variational principle is based on global maximizing the total complementary energy functional in terms of the stress field, satisfying the strain compatibility equation and the displacement boundary condition in a weak sense. The traction

* Corresponding author.
E-mail addresses: mechtb@uni-miskolc.hu (B. Tóth), mechklgy@uni-miskolc.hu (L.G. Kocsán).

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boundary condition, as well as the translational and rotational equilibrium equation are enforced in the strong sense using the second-order stress function tensor [23]. A dimensionally reduced cylindrical shell model based on this approach can be found in [24]. The related finite element scheme requires, however, C^1 continuous approximation of the second-order stress functions. This requirement, which is primarily caused by the *a priori* satisfaction of the symmetry condition for the stress tensor, makes it rather difficult and complicated to establish numerically efficient and well manageable elements for general shells.

One of the possibilities to avoid the difficulties mentioned in connection with development of numerically efficient stress based models is to incorporate the symmetry condition for the stress tensor into the total complementary energy functional using the rotations as Lagrangian multipliers. Applying this two-field dual-mixed variational principle, the translational equilibrium equations have to be satisfied *a priori* with the first-order stress function tensor. Plane elasticity models, as well as dimensionally reduced plate and shell models using the corresponding dual-mixed variational principle of Fraeijs de Veubeke [25] have been derived in [26–33]. The advantage of this principle is that the approximations of the first-order stress functions require only C^0 continuity between two elements.

The other alternative for the derivation of dimensionally reduced stress-based shell models is to ensure the satisfaction of only the translational equations in a weak sense, using the displacements as Lagrangian multipliers. Thus the two-field dual-mixed variational principle of Hellinger–Reissner [34–36] with *a priori* symmetric stresses can be obtained, where the displacement vector and the stress tensor are approximated as independent unknowns [37–45]. Although this approach usually yields good results for stresses, the development of stable and efficient finite element models has proven to be much more difficult than that with not *a priori* symmetric ones.

The lack of simple stable and efficient dual-mixed shell elements for the two-field Hellinger–Reissner functional has led to the construction of a modified Hellinger–Reissner variational principle in which the symmetry of the stresses is enforced weakly using the rotations as Lagrangian multipliers [46–52], retaining the basic stress and displacement variables. The resulting formulation has three independent fields: the stress tensor, the displacement vector and the rotation vector. Methods of this type are discussed in [6,53–57].

An outline of this paper is as follows. Dimensionally reduced cylindrical shell models are presented applying the two-field dual-mixed variational principle of Fraeijs de Veubeke and the three-field dual-mixed variational principle of Hellinger–Reissner. The functionals of the applied variational principles are described in Section 2. The geometric description of the cylindrical shell is given in Section 3.

A consistent dimensional reduction procedure is presented in Section 4. The approximation of the variables with respect to the thickness coordinate is given within Section 4.1. Then the number of the independent stress components is reduced by *a priori* satisfaction of the prescribed surface tractions on the inner and outer surfaces of the shell, assuming axisymmetrical loads in Section 4.2. A further reduction of the independent stress components is achieved by the elimination of certain rotations (Section 4.3). Then the self-equilibrated stress space of the Fraeijs de Veubeke variational principle is presented for cylindrical shells (Section 4.4).

In Sections 5.1 and 5.2, the first variation of the dual-mixed three-field Hellinger–Reissner and the two-field dual-mixed Fraeijs de Veubeke functionals are derived for thin cylindrical shells, applying the inverse stress–strain relations of linearly elastic, homogeneous and isotropic materials.

After choosing the applied polynomial spaces for *hp*-type approximations (Section 5.3), the computational performance of the finite element shell models are tested through two examples in Section 6. The convergence rates of the relative errors in energy norm, as well as in maximum norm of the directly approximated stresses are compared. The convergence curves for the relative errors measured in maximum norm of the displacements, approximated directly in the Hellinger–Reissner formulation, are also presented. The developed shell elements give very good numerical results for both *h*-extension and *p*-approximation independently of the thickness of the shell.

2. Dual-mixed functionals

The objective of this paper is to present similarities and differences of dimensionally reduced cylindrical shell models [33,57] based on dual-mixed variational principles using non-symmetric stresses. Throughout this paper, the usual summation convention is used and the range of the Latin indices is 1,2,3 and that of the Greek indices is 1 and 2. In the linear theory of elasticity their functionals can be derived from the total complementary energy functional

$$\mathcal{K}(\sigma^{pq}) = -\int_V U(\sigma^{pq})\,\mathrm{d}V + \int_{S_u} \tilde{u}_p \sigma^{pq} n_q\,\mathrm{d}S, \tag{1}$$

where V denotes the volume of the body bounded by the surface $S = S_p \cup S_u$, $(S_p \cap S_u = \emptyset)$, and $\tilde{u}_p$ is the displacement vector prescribed on the surface part S_u. The outward unit normal vector to the surface S is n_q. The complementary strain energy density function U can be expressed with the stress tensor σ^{pq} as

$$U(\sigma^{pq}) = \tfrac{1}{2}\sigma^{pq}\varepsilon_{pq}(\sigma^{rs}). \tag{2}$$

For linear elastic materials the symmetric strain tensor ε_{pq} can be obtained from the inverse stress–strain relations $\varepsilon_{pq} = C^{-1}_{pqrs}\sigma^{rs}$, where the fourth-order tensor C^{-1}_{pqrs} with symmetry properties $C^{-1}_{pqrs} = C^{-1}_{pqsr} = C^{-1}_{rspq}$ is the elastic compliance tensor. The subsidiary conditions to functional (1) are the translational equilibrium equations

$$\sigma^{pq}_{\cdot\cdot;q} + b^p = 0 \quad \text{in } V, \tag{3}$$

the rotational equilibrium equations

$$\epsilon_{pqr}\sigma^{pq} = 0 \quad \text{in } V, \tag{4}$$

and the stress boundary conditions

$$\sigma^{pq} n_q = p^p \quad \text{on } S_p, \tag{5}$$

where the subscript q preceded by a semicolon denotes the covariant derivative, b^p stand for the body forces, ϵ_{pqr} is the covariant permutation tensor, as well as p^p are the prescribed surface tractions on S_p.

To employ *a priori* non-symmetric stress field, the rotation equilibrium equation (4) is added to the functional (1) with Lagrangian multiplier technique. Thus we can obtain Fraeijs de Veubeke functional [30]

$$\mathcal{F}(\sigma^{pq}, \varphi^r) = -\int_V U(\sigma^{pq})\,\mathrm{d}V + \int_V \epsilon_{pqr}\sigma^{pq}\varphi^r\,\mathrm{d}V + \int_{S_u} \tilde{u}_p \sigma^{pq} n_q\,\mathrm{d}S, \tag{6}$$

where the infinitesimal rotation vector φ^r plays the role of the Lagrangian multiplier. The application of this principle requires the *a priori* satisfaction of the translational equilibrium equation (3) and the stress boundary conditions (5). A self-equilibrated stress field, i.e., a stress field that fulfills (3), can be obtained by