Finite Elements in Analysis and Design 97 (2015) 54–63

Contents lists available at ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.com/locate/finel

On the 'most normal' normal—Part 2

R. Aubry [a,*], E.L. Mestreau [a], S. Dey [b], B.K. Karamete [a], D. Gayman [a]

[a] Sotera Defense Solutions, 1501 Farm Credit Drive, Suite 2300, McLean, VA 22102-5011, USA
[b] US Naval Research Laboratory, Code 7131, 4555 Overlook Ave SW, Washington DC 20375, USA

ARTICLE INFO

Article history:
Received 2 April 2014
Received in revised form
16 January 2015
Accepted 26 January 2015
Available online 11 February 2015

Keywords:
Surface triangulation
Point normal
'Most normal' normal
Generalized Voronoi diagram on the sphere
Boundary layer mesh generation

ABSTRACT

In [1], a definition is given of the 'most normal' normal. This is the normal that minimizes the maximal angle with a given set of normals. The algorithm proposed does indeed compute a normal that verifies the previous property. However, this may not always be the 'most normal' normal. The previous normal should have more appropriately been called the 'most visible' normal. In the present work, an algorithm is designed to compute the real 'most normal' normal, namely the normal that maximizes the minimum angle with the planes carried by the triangles surrounding a point. This normal is the optimal point normal for boundary layer mesh generation if it is in the visibility cone. The algorithm consists in computing the generalized Voronoi diagram on the sphere of the edges created by the intersection between the triangles and the sphere. Numerical results illustrate the method, and compare with the previous algorithm.

1. Introduction

Surface triangulations are used to describe surfaces in many computer applications. Each triangle carries unambiguously a face normal, but some ambiguity may arise when defining a normal at the vertices or at the edges of the triangulation. Point normals are usually required in visualization [2], rendering [3], medical applications, boundary conditions in finite volume and finite elements solvers [4], free surface problems [5], coupled fluid-structure problems [6], tangent plane computations, curvature estimations [7,8], surface to surface interpolation [9], and so forth. One strategy to tackle the point normal computation is to consider the triangles as a discretization of a smooth surface. The problem therefore consists in providing a point normal as close as possible to the smooth surface normal. A different viewpoint is to consider the triangulation as a discrete object without specific connections to a smooth surface. In the context of a boundary layer mesh generator, prisms are extruded from the surface triangulation along the point normals [10–13]. An accurate computation of the normal is required, particularly along corners and ridges, since a bad normal may lead to a premature interruption of the boundary layer extrusion, a loss of orthogonality, or the creation of low quality elements.

In a previous work [1], a first definition of the 'most normal' normal was proposed. This normal minimizes the maximal angle with the face normals surrounding a given point. As noticed in Henrion et al. [14], this is one of many different definitions of a center of a cone. The algorithm proposed in the previous reference produces such a normal. Given the cone of normals emanating from the faces surrounding the vertex, the normal generated corresponds to the center of this cone. From a visibility standpoint, this normal is optimal by construction. For discrete remeshings, this provides a robust criterion to evaluate possible surface foldings [15,16]. It would therefore have been more appropriate to call it the 'most visible' normal. Fig. 1 shows nevertheless a drawback of this approach. Regardless of the local convexity or concavity, the same visibility cone is generated at the vertex. As noticed in [17], there is a duality between the plane carried by the triangles surrounding a vertex, and the face normals carried by these faces. In the projective plane [18], lines and points are dual objects, which can be interpreted on the sphere as a duality between three dimensional planes that contain the origin, and normals. Therefore, the dual strategy of minimizing the maximal angle between a vector and some face normals would be to maximize the minimum angle between this vector and the planes carried by the triangles. Fig. 2 depicts the same configurations seen from the dual viewpoint. While it seems similar in a two dimensional representation, it is not in three dimensions, since in two dimensions a point normal aligned on the bisector defines only two angles. Therefore minimizing the maximal angle is equivalent to maximizing the minimal angle.

The normality concept inherits from a scalar product definition. Since a scalar product also defines a distance, it is clear that there is an intimate relationship between normality and distance. For example, for a $G1$ surface [19], a tangent plane is available. Therefore the normal is computed as a normal to a plane, relying on the

* Corresponding author.
E-mail address: romain.aubry.ctr.fr@nrl.navy.mil (R. Aubry).

http://dx.doi.org/10.1016/j.finel.2015.01.005
0168-874X/

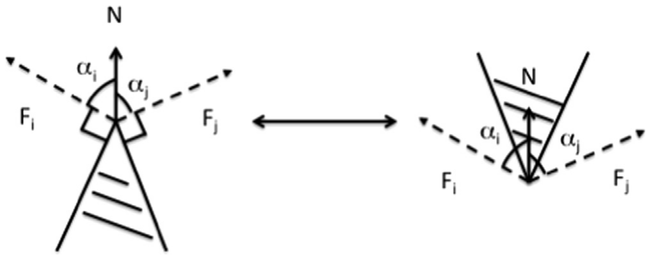

Fig. 1. The visibility cone is the same regardless of concavity or convexity.

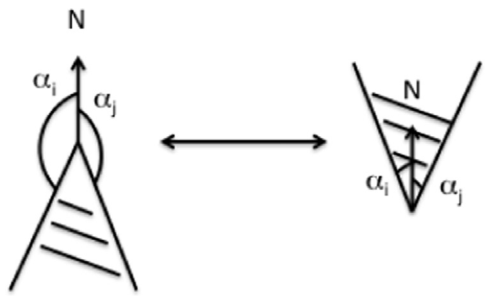

Fig. 2. The ‘most normal’ cone is not the same depending on concavity or convexity.

well-known cross product formula of the tangent vectors. However, degenerate patches may be created due to a particular parameterization. This does not mean that the normal does not exist. Some references [20,21] propose methods to compute normals for these situations. More generally, if a distance can be defined, a normal vector can also be obtained. A distance field is generated by resolving the Eikonal equation [22,23]. Normals are then obtained as a by product of the distance field through its gradient.

The organization of this paper is as follows. In Section 2, a short review of previously proposed normal computations is given. Then, Section 3 presents the algorithm to compute the ‘most normal’ normal, relying on the spherical generalized Voronoi diagram. Section 4 focuses in detail on the algebra and implementation of the spherical generalized Voronoi diagram. Finally, Section 5 illustrates the method through numerical examples.

2. Normal computation

Normal computation can be envisioned from different viewpoints depending on the application. As previously mentioned, a first viewpoint relies on the geometry of an underlying smooth surface that interpolates the vertices. When considering the point normal computation from an interpolation viewpoint, a generic strategy consists in taking a weighted average of all the face normals connected to a given point and normalizing it. With unit weights [2], a least-square approximation of the normal is sought. Some straightforward variations consider a different weighted average of the face normals according to Frey et al. [24]:

- weighted with the surface area of each triangle,
- weighted with the inverse of the surface area,
- weighted with the angle made by the two edges connected in the point under consideration.

Max [25] proposes a weighted sum of face normals, where the weights are computed in such a way that for a sphere surface, normals are exact regardless of the triangle size. Recently, Ubach et al. [26] propose a study of different point normal computations based on this strategy, as well as new weights formulae.

A second viewpoint may be based on a pure discrete setting. A direct application of this approach is boundary layer mesh generation, where the discrete surface is more relevant than the CAD smooth surface, since it will influence the quality of the generated elements. Within this context, the previous strategies may fail to produce either the ‘most visible’ normal or the ‘most normal’ normal because they are not associated with any theoretical optimum. Regarding the ‘most visible’ normal, a simple example is given by Pirzadeh in [27] where the trailing edge of a wing is meshed with four elements on the upper part and three on the lower part of the wing, producing an unbalanced normal in case of unit weights. A notable improvement was proposed in the context of RANS gridding by Kallinderis [11], based on the visibility cone of each point. The algorithm consists in choosing the pair of normals which produces the largest angle, taking a plane bisector of this faces, limiting the visibility on this plane by projecting the other faces on this plane, and finally taking the bisector of the angle created by the edges which limit the most the visibility in this plane. This normal is always valid if the faces allow it. Closely related, Pirzadeh in [27], proposed an iterative method relying on a predictor corrector basis. Finally, Aubry et al. [1] provide the optimum normal, as far as visibility is concerned.

A different approach was followed by Sani et al. [4], where normals are computed to satisfy the discrete incompressibility condition in a finite element context. Normals are then fully imposed on the mass conservation criterion, not considering the geometrical aspect of the boundary.

Finally, Wang et al. solve the Eikonal equation in [28] to offset the surface in the direction of the gradient of the solution. This work interestingly shifts the problem from a geometric triangle evaluation to a distance evaluation, where ultimately the normal is extracted as the gradient of the distance field. A similar approach is also proposed by Flin et al. [3], without relying on any triangulated surface.

3. The real ‘most normal’ normal

In this section, the main goal is to seek the normal that maximizes the minimal angle with the faces surrounding the vertex. Instead of computing vector to vector angles, vector to plane angles are involved, where all the vectors can be identified with a point on the unit sphere, and all the planes need only two points to be defined since they all contain the origin. The task is much more complex. Before, the ‘most visible’ normal was minimizing the maximum angle between normals, creating the largest cone that includes all normals. The dual strategy is equivalent to maximizing the minimum plane angle since the face and plane angles are linked through a $\alpha = (\pi/2) - \beta$ relationship. In terms of geodesic distance on the sphere, for the ‘most visible’ normal, the face normals were creating points on the unit sphere, and the goal was to obtain a point on the sphere that minimized the geodesic distance with all the other points. In this new approach, the new point should maximize the distance with the trace of the surrounding faces on the sphere, namely arcs of great circles. While the first method was closely related to the farthest Voronoi diagram of points, the new one consists in computing the generalized Voronoi diagram of arcs on the sphere. Every vertex of this generalized Voronoi diagram will be at the center of a small circle, therefore maximizing the geodesic distance. The possibly nonunique centers that have the maximum radius will define the ‘most normal’ normal.

In order to compute the generalized Voronoi diagram, one of the first methods in the plane is the one proposed by Lee [29]. Yao et al. [30] design a simpler algorithm to compute it. Regarding the Voronoi diagram on the sphere, Dinis et al. [31] and Zheng et al. [32] generalize the point Voronoi diagram to the sphere. Finally, Na et al. [33] propose to build an arbitrary spherical Voronoi diagram from two planar Voronoi diagrams of the same type and some clever merging of both