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Domain decomposition based finite element verification in linear framework

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ABSTRACT

The objective of this work is to obtain an *a posteriori* estimate of the discretization error of a reference problem. This reference problem is a mechanical problem solved using a finite element analysis in linear elasticity. We use the constitutive relation error based estimator. The construction of admissible fields, which is a pillar of the constitutive relation error method, is revisited using a domain decomposition method.

The originality of the present work is the introduction of a simplifying hypothesis that leads to an approximation of the discretization error. The construction of admissible fields, which is global, is then replaced by resolution of several local problems that conduce to lower CPU costs. The key point of this domain decomposition based error estimation is the fact that no inter-subdomain communication is needed. We present and illustrate this strategy and evaluate its benefits.

1. Introduction

In the field of mechanical engineering, numerical simulation has become an indispensable tool to predict the response of a structure to various solicitations. These numerical simulations involve the use of a discretized version of an initially continuous mathematical model (finite element analysis) that leads only to an approximation of the exact solution of the reference problem. In industrial situations, the Finite Element (FE) simulations have to satisfy both requirements of quality and CPU cost. On one side, quality requirement of the simulation, which is the purpose of verification consists in evaluating the distance between the exact solution and an approximated solution of the problem. A state-of-the-art review on verification can be found in [13,1]. On the other side recent development of parallel computers brought the foreground new simulations methods to achieve better CPU Cost. In particular, Domain Decomposition Method (DDM) is a more and more used strategy to take advantage of multicore processors or multiprocessor systems.

In Parret-Fréaud et al. [19], the proposed error estimator is strongly linked to the domain decomposition finite element based simulation, and is used to control the simulation. In this work, a new domain decomposition technique is introduced to compute the Finite Element analysis error, which takes a different path by defining totally independent calculations on each sub-domains. The main interest is a gain on CPU costs, the main drawbacks are a degraded error estimation, and thus a small quality loss. Note that the origin of the Finite Element solution can be varied. In particular, whether the finite element analysis can be performed elements using a domain decomposition method or not, the method developed here applies.

The objective of this paper is to illustrate that the quality loss is widely acceptable in a practical use. Section 2 reviews the equations of the reference problem and associated discretization and Section 3 reviews the constitutive relation error method and a technique for building admissible fields. Recovery of the admissible stress using a domain decomposition approach is introduced in Section 4. Sections 4 and 5 present some results obtained using these techniques.

2. Reference problem and its discretization

2.1. The reference problem

Let us consider a linear elastic static problem under the assumption of small transformations. The structure is defined in a domain Ω bounded by $\partial\Omega$. The structure is submitted to a prescribed displacement $\underline{u}_d$ over a part $\partial_1\Omega$ of the boundary $\partial\Omega$,

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a prescribed volume force $\underline{f}_d$ within Ω and a prescribed surface force density $\underline{F}_d$ over $\partial_2\Omega = \partial\Omega - \partial_1\Omega$.

Hence the reference problem: find the displacement field $\underline{u}(\underline{M})$ and the stress field $\sigma(\underline{M})$, defined at all points M of domain Ω, which satisfies

- the kinematic admissibility equations:

$$\underline{u} \in \mathcal{U}, \quad \underline{u} = \underline{u}_d \text{ on } \partial_1\Omega \tag{1}$$

- the static admissibility equations:

$$\sigma \in \mathcal{S}, \quad \sigma \cdot \underline{n} = \underline{F}_d \text{ on } \partial_2\Omega \tag{2}$$

$$\sigma \in \mathcal{S}, \quad \underline{div}(\sigma) + \underline{f}_d = \underline{0} \text{ on } \Omega \tag{3}$$

- and the constitutive relation:

$$\sigma = \mathbf{K}\epsilon(\underline{u}) \text{ on } \Omega \tag{4}$$

where $\mathcal{U}$ is the set of the fields $\underline{u}$ which are regular, and $\mathcal{S}$ is the set of the fields σ which are symmetrical and regular (*i.e.* with finite energy). $\epsilon(\underline{u})$ represents the linearized strain tensor: $\epsilon(\underline{u}) = 1/2(\underline{grad}(\underline{u}) + \underline{grad}(\underline{u})^t)$. Hooke's tensor is denoted $\mathbf{K}$. Now let $\mathcal{U}_{ad,0}$ be the space of the fields which are kinematically admissible to zero:

$$\mathcal{U}_{ad,0} = \{\underline{u} | \underline{u} \in \mathcal{U}, \ \underline{u}|_{\partial_1\Omega} = \underline{0}\} \tag{5}$$

and $\mathcal{S}_{ad,0}$ the space of the fields which are statically admissible to zero:

$$\mathcal{S}_{ad,0} = \{\sigma | \sigma \in \mathcal{S}, \underline{div}(\sigma) = \underline{0}, \ \sigma|_{\partial_2\Omega} \cdot \underline{n} = \underline{0}\} \tag{6}$$

2.2. The discretized problem

In practice, the exact solution of the reference problem $(\underline{u}_{ex}, \sigma_{ex})$ is often unknown. Therefore, one calculates an approximate solution by introducing a discretized problem based on a weak formulation of the model's equations. This leads to an approximated pair $(\underline{u}_h, \sigma_h)$ which is the solution of the discretized problem. This solution is defined from a set of approximate displacements $\mathcal{U}_h \subset \mathcal{U}$.

Thus, the finite element problem consists in seeking the pair $(\underline{u}_h, \sigma_h)$ which satisfies

- the kinematic admissible equations:

$$\underline{u}_h \in \mathcal{U}_h, \quad \underline{u}_h|_{\partial_1\Omega} = \underline{u}_d, \tag{7}$$

- the equilibrium equation:

$$\sigma_h \in \mathcal{S}, \quad \int_\Omega \mathrm{Tr}[\sigma_h \cdot \epsilon(\underline{u}_h^*)]\, d\Omega = \int_\Omega \underline{f}_d \cdot \underline{u}_h^*\, d\Omega + \int_{\partial_2\Omega} \underline{F}_d \cdot \underline{u}_h^*\, d\Omega, \quad \forall \underline{u}_h^* \in \mathcal{U}_{ad,0} \cap \mathcal{U}_h, \tag{8}$$

- and the constitutive relation:

$$\sigma_h = \mathbf{K}\epsilon(\underline{u}_h). \tag{9}$$

3. Error in constitutive relation and associated admissible construction

3.1. The error in constitutive relation

The difference between the solution of the finite element problem $(\underline{u}_h, \sigma_h)$ and the exact solution of the mathematical problem $(\underline{u}_{ex}, \sigma_{ex})$ is called the discretization error. This error is due mainly to the approximation concerning the equilibrium equations, the spatial discretization of the geometry or the approximation concerning the loading.

For the purpose of dimensioning the structure, it is important to quantify that error and, if possible, to obtain an upper bound. A global measure of the error is defined by

$$e = \| \sigma_{ex} - \sigma_h \|_{\mathbf{K}^{-1},\Omega}, \tag{10}$$

where σ_{ex} is the stress field which is the exact solution of the reference problem and σ_h is the approximated stress field obtained using the finite element method. The energy norm is based on Hooke's tensor $\mathbf{K}$ on Ω:

$$\| \bullet \|^2_{\mathbf{K}^{-1},\Omega} = (\bullet, \bullet)_{\mathbf{K}^{-1},\Omega} = \int_\Omega \mathrm{Tr}[\bullet \mathbf{K}^{-1} \bullet]\, d\Omega. \tag{11}$$

To estimate e, we use an error estimator based on the constitutive relation error (CRE), see Ladevèze and Pelle [13]. A measure of the non-verification of the constitutive relation by an admissible pair $(\hat{\underline{u}}, \hat{\sigma})$ is introduced:

$$e^2_{CRE}(\hat{\underline{u}}, \hat{\sigma}) = \| \hat{\sigma} - \mathbf{K}\epsilon(\hat{\underline{u}}) \|^2_{\mathbf{K}^{-1},\Omega}. \tag{12}$$

A pair $(\hat{\underline{u}}, \hat{\sigma})$ is said to be admissible if $\hat{\underline{u}}$ satisfies Eq. (1) and if $\hat{\sigma}$ satisfies Eqs. (2) and (3). Since field $\underline{u}_h$ satisfies Eq. (7), it necessarily satisfies Eq. (1); therefore, one generally chooses $\hat{\underline{u}} = \underline{u}_h$. The construction of field $\hat{\sigma}$ is reviewed in Section 3.2. The error in constitutive relation satisfies the Prager Synge relation Prager and Synge [21] which leads to the following upper bound:

$$e \le e_{CRE}. \tag{13}$$

This upper bound of the global error can be used to obtain bounds of the local quantities thanks to the introduction of a dual problem, see Becker and Rannacher [3]. The contributions of this error can also be used to perform an adaptative mesh refinement, see Florentin et al. [5].

3.2. Construction of the statically admissible stress field

While the admissible displacement field construction is straight forward, the construction of $\hat{\sigma}$ requires a specific recovery step based on the finite element solution σ_h and the data of the reference problem. Different paths can be used see e.g. [18,19,16,9]. The construction we use here consists of two steps:

- Construction of nodal densities $\hat{\underline{F}}_h$
  The nodal densities $\hat{\underline{F}}_h$ are defined along the edge Γ belonging to ∂E, the set of the edges of each element. These densities are calculated to be in equilibrium with the problem's boundary conditions, *i.e.* they satisfy:

$$\hat{\underline{F}}_h = \underline{F}_d \quad \text{on} \quad \partial_2\Omega \tag{14}$$

$$\int_E \underline{f}_d \cdot \underline{U}_h^* dE + \int_{\partial E} \eta_E^\Gamma \hat{\underline{F}}_h \cdot \underline{U}_h^*\, d\Gamma = 0, \quad \forall \underline{U}_h^* \text{ solid body motion of } E. \tag{15}$$

  In order to guarantee the continuity of the loads along edge Γ associated to E, we have introduced $\eta_E^\Gamma = \pm 1$, with $\eta_{E_i}^\Gamma \cdot \eta_{E_j}^\Gamma = -1$ for two adjacent elements E_i and E_j, see Ladevèze and Pelle [13] for more details. Field $\hat{\underline{F}}_h$ can be built in several ways. For this work, we chose what is called the “optimal method”, which consist in a global minimization problem defined on Ω, see [6] for more details.
- Construction of $\hat{\sigma}$ from $\hat{\underline{F}}_h$
  Once field $\hat{\underline{F}}_h$ has been calculated, the field of the admissible stresses is built element by element. Analytical methods can be used, see Ladevèze and Pelle [13]. One can also use a numerical