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An Element-free Galerkin (EFG) scaled boundary method

Yiqian He [a], Haitian Yang [a], Andrew J. Deeks [b,*]

[a] State Key Lab of Structural Analysis for Industrial Equipment, Department of Engineering Mechanics, Dalian University of Technology, Dalian 116024, PR China
[b] School of Engineering and Computing Sciences, Durham University, South Road, Durham, DH1 3LE, UK

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ABSTRACT

The scaled boundary method is a semi-analytical method of analysis which can be used in computational mechanics. This method uses a normalised radial coordinate system, introducing shape functions to weaken the governing equations in the circumferential direction and solving the resulting differential equations analytically in the radial direction. This paper presents a new Element-free Galerkin (EFG) scaled boundary method in which the EFG approach is used in the circumferential direction. The proposed model is verified by application to a number of standard problems of elasticity. The numerical solutions show that the new method has higher accuracy (for any particular number of nodes) and better convergence than scaled boundary finite element methods, and an accurate smooth stress field can be obtained directly without the necessity of using a stress recovery procedure.

1. Introduction

The scaled boundary method is a semi-analytical method developed relatively recently by Wolf and Song [1]. The method introduces a normalised radial coordinate system based on a scaling centre and a defining curve (usually taken as the boundary). The governing deferential equations are weakened in the circumferential direction and then solved analytically in the normalised radial direction. Like the boundary element method, discretisation of the boundary only is required, but unlike that method no fundamental solution is required. The method has been shown to be more efficient than the finite element method for problems involving unbounded domains and for problems involving stress singularities or discontinuities [2]. Especially, fruitful applications of this method have been achieved for fracture problems [3–6] and foundation problems [7–10].

In the scaled boundary method, the discretisation approach used in the circumferential direction has significant influence on the accuracy of the resulting solutions [11]. The most commonly used method for performing this circumferential discretisation is the finite element approach, leading to the method called the scaled boundary finite element method (SBFEM) [12–16]. However, like the finite element method, this approach results in the computed stress field being discontinuous between nodes, and necessitates use of stress recovery procedures, such as those reported by Deeks and Wolf [2]. The development of meshless methods provided another approach to build circumferential approximations for the scaled boundary method. Deeks and Augarde [11] presented a meshless version of the scaled boundary method based on a Meshless Local Petrov–Galerkin approach. They showed that the MLPG scaled boundary method gave a higher level of accuracy and rate of convergence than the conventional SBFEM using quadratic elements. However, as bell-shaped weight functions are used for the test functions, the approach results in a more complex formulation of the scaled boundary equations leading to an unsymmetrical stiffness matrix. In addition, the authors experience with the MLPG scaled boundary method has shown that the method is not robust in certain situations, due to the bell-shaped weight functions not summing to unity everywhere in the domain. This causes the error in certain parts of the domain to be over-weighted, and the error in other parts of the domain to be under-weighted, sometimes leading to sub-optimal solutions.

The original EFG method was developed by Belytschko et al. [17]. This method is only based on nodes, and thus no mesh generation or remeshing is required. It has been shown that, compared with the finite element method, the EFG method has the advantages of high accuracy, rapid convergence, and a smooth stress solution can be obtained without post-processing [17]. To the authors' knowledge, there has been no work done which combines the EFG method and the scaled boundary method.

Therefore it is attractive to establish a new Element-free Galerkin scaled boundary method (EFG–SBM) combining EFG and scaled boundary method, in which the Moving Least Square (MLS) shape functions are used in the circumferential direction of the scaled boundary model based on the Galerkin approach. The incorporation of the EFG approach into the scaled boundary method combines the advantages of scaled boundary method and

* Corresponding author.
E-mail address: a.j.deeks@durham.ac.uk (A.J. Deeks).

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EFG method in a new EFG–SBM. This new numerical model has the advantages that

(1) Discretisation is only required in the circumferential direction, so the number of degrees of freedom (DOF) required to solve a problem is significantly reduced compared with the conventional EFG method in a 2D domain, but, unlike the conventional boundary element method, no the fundamental solution is required and no integrals containing singularities need be performed.
(2) The EFG–SBM provides a new approach to establishing a meshless boundary method. Other meshless boundary methods described in the literature are mainly based on Boundary Integral Equations (BIE), including the Local Boundary Integral Equation (LBIE) method [18], the Boundary Element-free Method (BEFM) [19–21], and the Boundary Node Method (BNM) [22]. Compared with these BIE based meshless methods, the EFG–SBM does not use the BIE approach, and so no fundamental solution is required.
(3) The solutions are C^1 continuous over the domain, and so smooth stress fields can be obtained without post-processing, and the advantages of accuracy, smoothness and convergence of original EFG can be retained.
(4) Like the original SBFEM, this meshless version of the scaled boundary model is also suitable for solving problems involving unbounded domains and stress singularities or discontinuities very efficiently.
(5) In comparison to the MLPG scaled boundary method, the EFG approach yields a symmetric stiffness matrix and a more robust solution, since in the MLPG approach the weighing functions do not comply with a partition of unity approach, while in the EFG approach they do.
(6) Since the symmetry of original scaled boundary method is retained, the method can be combined with domains modelled with the finite element method or the EFG method much more conveniently than with the MLPG scaled boundary method.

This paper is organised as follows: The basic equations of scaled boundary method are given in the next section. Section 3 introduces an EFG approach for scaled boundary methods. Some example problems are presented in Section 4 to verify the effectiveness of proposed method, and the paper draws some conclusions at the end. Since the computational advantage of the SBFEM over the finite element method for problems involving stress singularities, stress discontinuities and unbounded domains has been demonstrated previously [2], this paper only compares the performance of this EFG–SBM method with the existing SBFEM and the MLPG scaled boundary method.

2. The scaled boundary method

The scaled boundary method introduces a normalised radial coordinate system by scaling the domain boundary relative to a scaling centre (x_0,y_0) selected within the domain (Fig. 1). The normalised radial coordinate ξ runs from the scaling centre towards the boundary, and has values of zero at the scaling centre and unity at the boundary, which can be considered as the defining curve for the coordinate system. The other circumferential coordinate s specifies a distance around the boundary from an origin on the boundary. The scaled boundary and Cartesian coordinate systems are related by the scaling equations

$$x = x_0 + \xi x_s(s) \tag{1}$$

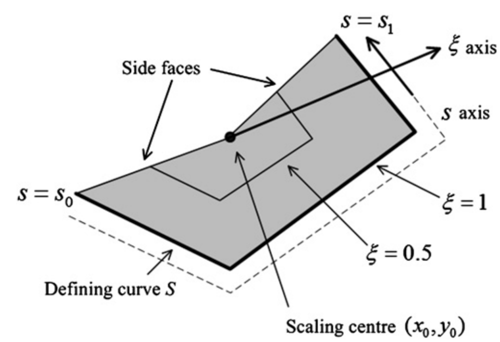

Fig. 1. Bounded domain with side faces showing scaled boundary coordinate system.

$$y = y_0 + \xi y_s(s) \tag{2}$$

Displacement and stress components are retained in the original Cartesian coordinate directions, while position is specified in terms of the scaled boundary coordinates. An approximate solution is sought in the form

$$\{u_h(\xi,s)\} = \sum_{i=1}^{n} [N_i(s)] u_{hi}(\xi) = [N(s)]\{u_h(\xi)\} \tag{3}$$

This represents a discretisation of the boundary $\xi = 1$ with the shape function $[N(s)]$. The unknown vector $\{u_h(\xi)\}$ is a set of n functions analytical in ξ. The shape functions apply for all lines with a constant ξ.

Mapping the linear operator to the scaled boundary coordinate system using standard methods

$$[L] = \left[L^1\right]\frac{\partial}{\partial x} + \left[L^2\right]\frac{\partial}{\partial y} = \left[b^1(s)\right]\frac{\partial}{\partial \xi} + \frac{1}{\xi}\left[b^2(s)\right]\frac{\partial}{\partial s} \tag{4}$$

where $[b^1(s)]$ and $[b^2(s)]$ are dependent only on the boundary definition.

The stresses are obtained by multiplying the strains (obtained form the displacement field using the linear operator) by the elasticity matrix $[D]$ in the form

$$\{\sigma(\xi,s)\} = [D]\{\varepsilon(\xi,s)\} = [D]\left[B^1(s)\right]\{u(\xi)\}_{,\xi} + \frac{1}{\xi}[D]\left[B^2(s)\right]\{u_h(\xi)\} \tag{5}$$

where

$$[B^1(s)] = [b^1(s)][N(s)] \tag{6}$$

$$[B^2(s)] = [b^2(s)][N(s)]_{,s} \tag{7}$$

In this case the virtual work statement becomes

$$\int_V \{\delta\varepsilon(\xi,s)\}^T\{\sigma_h(\xi,s)\}dV - \int_S \{\delta u(s)\}^T\{t(s)\}\mathrm{d}s = 0 \tag{8}$$

where the first term represents the internal work and the second term the external work, and $\{t(s)\}$ is the external force vector.

The virtual strain field is of the form (analogous to Eq. (5))

$$\{\delta\varepsilon(\xi,s)\} = \left[B^1(s)\right]\{\delta u(\xi)\}_{,\xi} + \frac{1}{\xi}\left[B^2(s)\right]\{\delta u(\xi)\} \tag{9}$$

where $\{\delta u(\xi)\}$ is virtual displacement and

$$\mathrm{d}V = |J|\xi \mathrm{d}\xi \mathrm{d}s \tag{10}$$

where $|J|$ is the Jacobian at the boundary $(\xi = 1)$.

Substituting Eqs. (5), (9) and (10), integrating the area integrals containing $\{\delta u(\xi)\}_{,\xi}$ with respect to ξ using Green's Theorem, and introducing the coefficient matrices

$$[E^0] = \int_S [B^1(s)]^T[D][B^1(s)]|J|\,\mathrm{d}s \tag{11}$$