Finite Elements in Analysis and Design 120 (2016) 1–17

Contents lists available at ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.com/locate/finel

Discontinuous Galerkin method with arbitrary polygonal finite elements

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ARTICLE INFO

Article history:
Received 22 September 2015
Received in revised form
6 May 2016
Accepted 6 June 2016

Keywords:
Discontinuous Galerkin method
Polygonal finite elements
Finite difference
Elliptic problem

ABSTRACT

The paper presents the efficient application of discontinuous Galerkin (DG) method on polygonal meshes. Three versions of the DG method in which the approximation is constructed using sets of arbitrary basis functions are under consideration. The analysed approach does not require definition of nodes or construction of shape functions. The shape of a polygonal finite element (FE) can be quite arbitrary. It can have arbitrary number of edges and can be non-convex. In particular, a single FE can have a polygonal hole or can even consist of two or more completely separated parts. The efficiency, flexibility and versatility of the presented approach is illustrated with a set of benchmark examples. The paper is restricted to two-dimensional case. However, direct extension of the algorithms to three-dimensions is possible.

1. Introduction

This paper considers the discontinuous Galerkin (DG) method for polygonal meshes. The application of DG method on the polygonal meshes is currently quite a new topic and there are a few papers tackling this problem, e.g. [1–7]. In this work, the DG method on the polygonal meshes is extended to the cases in which the finite element cells have arbitrary shape and can be convex or non-convex. Here, the cells can be non-simply-connected (NSC) (i.e. a cell with holes inside) or even non-connected (NC) (i.e. a cell consisting of two or more parts which are completely separated from each other). The approximation in the finite element cells is based on a set of basis functions which can consist of quite arbitrary functions. It means that there are no special requirements for those functions on edges or vertices of the cell.

Most discrete methods apply the standard meshes, that is: triangular and quadrilateral meshes in two-dimensional case, or tetrahedral, hexahedral, prismatic and pyramidal meshes in three dimensions. The direct extension of standard meshes are polygonal or polyhedral meshes, respectively. Such meshes provide new flexibility in domain discretisations, especially in complex geometries, incorporating periodic boundary conditions or problems with specific physical constraints [8], e.g. modelling a composite microstructure [9]. On the other hand, all the properties of a polygonal and a polyhedral are not yet well known and the research concerning such meshes is needed. In this paper, a comparison of results gained on polygonal mesh with these on triangular mesh is presented and it can be concluded that the polygonal mesh provides better results. Although, this work deals with the problem of polygonal finite elements, some conclusions can be directly extended to polyhedral elements.

Although, the combination of DG method with polygonal meshes is quite new, the idea of polygonal finite element method (PFEM) reaches 1975 when Wachspress in [10] introduced the barycentric coordinates and barycentric interpolation and a rational basis on convex polygons. The renaissance of the Wachspress work took place at the beginning of our century when the Wachspress basis functions have been applied to finite element method (FEM) [11,12]. Afterwards, a series of other papers have appeared concerning the FEM on polygonal or polyhedral meshes, e.g. [13–24]. There are two other methods that deal with polygonal/polyhedral meshes i.e. mimetic finite difference method (MFD) (e.g. [25–27]) and virtual element method (VEM) (e.g. [28–33]). A reliable survey on all the methods based on polygonal/polyhedral meshes is presented in [34]. The computer applications of the mentioned methods, namely PFEM, MFD and VEM are not simple since they either require complicated shape functions or complex algorithm to be implemented. To apply such methods a great computational effort has to be made. A competitive alternative to these methods is the approaches based on DG idea. They do not require shape functions construction or any other sophisticated problem formulation.

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http://dx.doi.org/10.1016/j.finel.2016.06.004
0168-874X/

In this paper the approximation on finite elements is based on a set of basis functions, i.e. a set of monomials. In consequence, no shape functions or nodes are applied in such an approach. Therefore, the approximation can be easily applied to finite element cell with arbitrary shape and furthermore, the elements can have holes and can consist of two or more completely separated parts. The DG methods in this paper are based on the standard Galerkin weak formulation and give great flexibility in selecting the approximation order or the finite element shapes, which can be quite arbitrary. In the DG method, for two-dimensional case, the same algorithm is applied to standard triangular or quadrilateral meshes as well as to meshes with convex and non-convex, NSC and NC polygons. The integration has to be performed over all finite elements, outer boundary and over the mesh skeleton (the finite elements boundaries). The integration over a finite element is performed in such a way that its region is subdivided into a set of triangles and then the standard Gauss method is applied to each of them.

Here, three versions of DG method are considered:

1. discontinuous Galerkin with finite difference (DGFD) method [35],
2. interface discontinuous Galerkin (IDG) method [36],
3. standard discontinuous Galerkin (SDG) method [37].

In all the versions of the DG method the approximation fields are constructed by a set of local basis functions in the same way as in [35]. The results obtained from all three methods are presented and compared.

The remaining part of this paper is organised as follows: Section 2 defines the elliptic mathematical problem for which the DG methods are derived. The discontinuous Galerkin formulations on polygonal meshes are presented in Section 3 where the IDG, DGFD and SDG methods are described in Sections 3.1, 3.2 and 3.3, respectively. Section 4 deals with the approximation techniques where arbitrary basis functions can be used. The DG methods are illustrated with a series of benchmark examples in Section 5. The paper finishes with final conclusions.

2. Mathematical model

The DG methods are presented for scalar elliptic problem that has a physical interpretation of the stationary heat transport. The problem starts with the well-known local form of energy balance equation, Fourier's law as well as boundary conditions of Dirichlet and Neumann types. The problem is defined in the domain Ω with outer boundary Γ as follows: find the continuous temperature field Θ that satisfies the following relations:

$$\begin{aligned} &\operatorname{div}\mathbf{q}-r=0, \quad \text{in } \Omega \\ &\mathbf{q}=-k\nabla\Theta, \quad \text{in } \Omega \\ &\Theta=\hat{\Theta} \quad \text{on } \Gamma_\Theta \\ &\mathbf{q}\cdot\mathbf{n}=\hat{h} \quad \text{on } \Gamma_q \end{aligned} \tag{1}$$

where $\mathbf{q}$ is the heat flux vector, r is the heat source density, k is the heat conductivity parameter for a thermally isotropic material, $\hat{\Theta}$ and $\hat{h}$ are prescribed values of temperature and heat flux, respectively, Γ_Θ is the part of Γ in which the temperature $\hat{\Theta}$ is prescribed, Γ_q is the part of Γ in which the heat flux $\hat{h}$ is prescribed and $\mathbf{n}$ is the unit vector normal to the outer boundary. The heat flux vector is related to the temperature field by Fourier's law in $(1)_2$.

The regarded domain is structured by polygonal mesh in 2D. The mesh consists of a set of cells, outer boundary and the mesh skeleton, illustrated in Fig. 1. Individual cell is a polygon which may not necessarily be convex, simply-connected or connected. The DG method requires integration along the mesh skeleton. In this paper such integration is performed with the help of the skeleton local coordinates. The local coordinates are defined by a set of mutually perpendicular unit vectors $\mathbf{n}^s, \mathbf{s}^s$ where vector $\mathbf{n}^s$ is normal to the skeleton and referred to as the skeleton normal and $\mathbf{s}^s$ is the skeleton tangent. The orientation of these vectors is arbitrary, providing that they meet the above requirements. It means that the skeleton normal can be directed to either of the aligned cells, the same refers to the skeleton tangent. Fig. 2 shows the set of the skeleton local coordinates for a polygonal mesh.

The DG method is based on the discontinuous approximation with the discontinuity on the skeleton. The discontinuity has to be regarded in the global formulation of the problem defined by Eq. (1), where the integration by parts is performed. Consequently, it leads to the situation when the jump and mean values of the discontinuity have to be regarded. Their definitions are based on the skeleton normal, namely

$$[\![f]\!]=\lim_{\epsilon\to 0}[\![f]\!]_\epsilon, \quad \langle f\rangle(\mathbf{x})=\lim_{\epsilon\to 0}\langle f\rangle_\epsilon \tag{2}$$

where the jump and mean values at distance ϵ are defined as follows:

$$[\![f]\!]_\epsilon=f(\mathbf{x}+\epsilon\mathbf{n}^s)-f(\mathbf{x}-\epsilon\mathbf{n}^s), \quad \langle f\rangle_\epsilon=0.5\left(f(\mathbf{x}+\epsilon\,\mathbf{n}^s)+f(\mathbf{x}-\epsilon\,\mathbf{n}^s)\right) \tag{3}$$

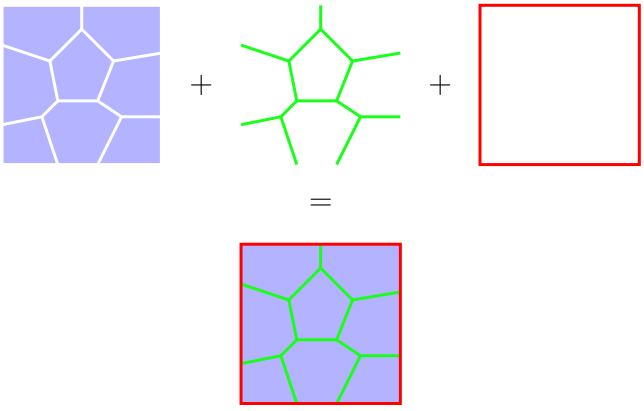

Fig. 1. Polygonal mesh consisting of a set of cells, outer boundary and mesh skeleton.

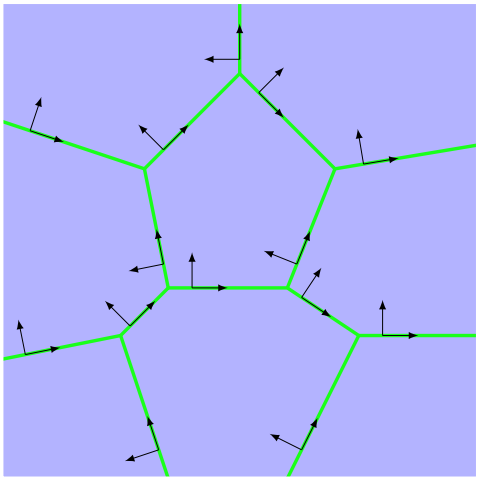

Fig. 2. Mesh skeleton with a set of skeleton local coordinates: normal and tangent.