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Blending isogeometric and Lagrangian elements in three-dimensional analysis

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ABSTRACT

A method for blending three-dimensional Non-Uniform Rational B-Splines (NURBS) elements and Lagrangian elements is proposed. In the blended element, selected boundary facets of a NURBS patch are presented in Lagrangian form, enabling the patch to be directly connected to standard Lagrangian elements. The transformation from NURBS to Lagrangian preserves the original geometry and thus retains the geometric exactness. The transformation is utilized to develop blended models wherein a part of the domain is described in NURBS while the remaining domain is in traditional finite elements. A simply supported beam is used as the benchmark example to test the convergence of the 3D blended element model. The utility of the blended model is also demonstrated with an example of a gear with intricate geometry. The stress solutions are compared with those of the pure finite element model and the validity of the blended analysis is demonstrated.

1. Introduction

Finite element method (FEM) has been the most widely used numerical method in engineering analysis. In FEM a material body is represented by a topological mesh which discretizes the body into a collection of elements. The mesh is an additional geometric entity, and one that is disconnected from an underlying geometric model. To better integrate geometric modeling and analysis, Hughes [1] put forward the concept of isogeometric analysis (IGA). The basic idea of the IGA is to unify the geometric basis, using the geometric language for both design and analysis. Currently the NURBS [2] and T-Splines [3] are the primary geometry languages used in isogeometric analysis [4–6]. NURBS is the de-facto standard in Computer Aided Design (CAD); thus using NURBS enables a direct integration between CAD and analysis. Compared with the standard FEM, the IGA has some remarkable merits [7–9]: (1) It provides a more accurate description for complicated geometries, and in particular, common shapes such as circle, cylinder, sphere, and ellipsoid can be represented exactly [10,11]. (2) The original geometry is preserved in h- p- and k-refinement [12]. (3) It offers up to C^{p-1} continuity between elements [13], in contrast to the C^0 continuity in FEM, and (4) it facilitates an integration of analysis, design, and optimization at the geometry level.

While the theoretical foundation of IGA has been firmly established, the development of IGA computer code, especially general purpose programs such as existing ones in FEM, is still at the early stage. One of hurdles has been to automatically convert a 3D CAD model into NURBS meshes. While significant advances have been made in recent years [14–17], generating NURBS meshes for complicated 3D geometries remains a challenge. We envision that, for the time being there is a need for *blended models* which admit both traditional Lagrangian elements and NURBS/T-spline isogeometric elements. Conceivably, for some problems it is unnecessary to describe the entire domain in NURBS or T-Splines. One could leave a part of the domain in FEM while describing some crucial regions in NURBS. Similar situations can arise in the analysis of multi-component structures where some components are supplied in FE meshes while others are in NURBS. It would be convenient to keep both representations in the analysis, giving rise to a blended model. This blended modeling enables the advantages of both methods to be exploited. Perhaps more importantly, it enables one to incorporate isogeometric features in well-developed finite element programs.

A key step in blending Lagrangian and NURBS elements is to develop interface elements containing mixed representations. Fortunately, interfacing NURBS to Lagrangian elements is straightforward, as both are polynomial based representations. B-spline basis functions and Lagrangian polynomials are mutually expressible because both are polynomial basis. Based on this observation, Lu et al. [18] proposed a blending technique; the basic idea is to represent some edges or facets of a NURBS element in

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Lagrangian form. However, in our previous 2D model we have not discussed the blended-element (a quadratic NURBS element with a quadratic Lagrangian element and a quadratic NURBS element with a linear finite element) simulation. In addition, three dimensional models have not been developed. In [18] the parent elements are in NURBS. A NURBS-enhanced finite element method (NEFEM) was proposed by Sevilla et al. [19]. There the parent element was Lagrangian, and selected boundaries were transformed into NURBS. A similar setting was adapted by Corbett et al. [20] in contact analysis where NURBS representation was used on contact surfaces to improve the smoothness of geometric representation. In a different front, efforts have been devoted to coupling isogeometric analysis and meshfree methods [21–23]. The development in [18] and similar contributions hinges on the idea of inter-representation between splines and other polynomial basis. In isogeometric analysis, this idea can be traced back to [24,25]. Therein the authors introduced the concept of Bézier extraction which represents each element (a non-empty knot span or tensor product of such) in terms of Bernstein polynomials defined over the element. A Bézier element contains a fixed set of basis functions, just like regular Lagrangian elements. Recently, the concept of Lagrange extraction was proposed by Schillinger et al. [26] representing all elements in a NURBS domain in Lagrangian form. This establishes a direct link between splines and nodal finite element basis functions.

In this paper, we will develop the 3D blended isogeometric and Lagrangian elements. We will introduce the geometric representation of blended elements, and discuss how much representations are constructed. While the essential idea remains the same as the previous work in 2D, the details of the geometric representation need to be worked out. This paper is organized as follows: in Section 2 the geometric basis for transforming a NURBS curve or surface into a Lagrangian form is reviewed. The construction of three dimensional blended elements, including the treatment of interfacing elements of different polynomials degrees, is presented in Sections 3 and 4. Numerical examples are introduced in Section 5 to demonstrate the application of the proposed approach.

2. Lagrangian transformation

2.1. NURBS curve

A p-order NURBS curve is specified by a knot vector $U=\{u_1,u_2,\dots,u_{n+p+1}\}$, a set of control points and corresponding weights $\{\mathbf{Q}_i,\omega_i\}_{i=1}^{n}$. The curve is expressed as

$$\mathbf{C}(u)=\frac{\sum_{i=1}^{n}N_{i,p}(u)\omega_i\mathbf{Q}_i}{\sum_{i=1}^{n}N_{i,p}(u)\omega_i} \tag{1}$$

where $N_{i,p}(u)$ are the B-spline basis functions that are determined by the knot vector and the degree p.

A NURBS curve is a composition of polynomial segments (or elements), each corresponding to a non-empty knot interval. A polynomial segment can be represented using Lagrangian interpolation. To this end, consider the segment $[u_k,u_{k+1}](k\ge p+1)$ and introduce $p+1$ interpolation points $\overline{u}_{l+i}=(1-i/p)u_k+(i/p)u_{k+1},i=0,\dots,p$. These points determine a set of Lagrangian basis $\{L_{i,p}\}_{i=l}^{l+p}$. We seek to express the segment in a rational Lagrangian form

$$\mathbf{C}(u)=\frac{\sum_{i=l}^{l+p}L_{i,p}(u)\varpi_i\boldsymbol{P}_i}{\sum_{i=l}^{l+p}L_{i,p}(u)\varpi_i} \tag{2}$$

where $\{\boldsymbol{P}_i\}_{i=l}^{l+p}$ are Lagrangian nodes and $\{\varpi_i\}_{i=l}^{l+p}$ are the corresponding weights. The Lagrangian nodes and weights are determined by setting the numerators and denominators to be equal in both forms:

$$\sum_{i=k-p}^{k}N_{i,p}(u)\mathbf{Q}_i^{\omega}=\sum_{i=l}^{l+p}L_{i,p}(u)\boldsymbol{P}_i^{\varpi}$$
$$\sum_{i=k-p}^{k}N_{i,p}(u)\omega_i=\sum_{i=l}^{l+p}L_{i,p}(u)\varpi_i \tag{3}$$

where $\mathbf{Q}_i^{\omega}=\omega_i\mathbf{Q}_i$ and $\boldsymbol{P}_i^{\varpi}=\varpi_i\boldsymbol{P}_i$.

Due to the interpolation property of the Lagrangian basis, the nodes $_{Pi}$ is nothing but the value of curve at $\overline{u}_i$. It follows that

$$\begin{bmatrix}\boldsymbol{P}_l^{\varpi}\\ \boldsymbol{P}_{l+1}^{\varpi}\\ \vdots\\ \boldsymbol{P}_{l+p}^{\varpi}\end{bmatrix}=\mathbf{M}\begin{bmatrix}\mathbf{Q}_{k-p}^{\omega}\\ \mathbf{Q}_{k-p+1}^{\omega}\\ \vdots\\ \mathbf{Q}_k^{\omega}\end{bmatrix} \tag{4}$$

$$\begin{bmatrix}\varpi_l\\ \varpi_{l+1}\\ \vdots\\ \varpi_{l+p}\end{bmatrix}=\mathbf{M}\begin{bmatrix}\omega_{k-p}\\ \omega_{k-p+1}\\ \vdots\\ \omega_k\end{bmatrix} \tag{5}$$

where

$$\mathbf{M}=\begin{bmatrix}N_{k-p,p}(\overline{u}_l) & N_{k-p+1,p}(\overline{u}_l) & \cdots & N_{k,p}(\overline{u}_l)\\ N_{k-p,p}(\overline{u}_{l+1}) & N_{k-p+1,p}(\overline{u}_{l+1}) & \cdots & N_{k,p}(\overline{u}_{l+1})\\ \vdots & \vdots & \vdots & \vdots\\ N_{k-p,p}(\overline{u}_{l+p}) & N_{k-p+1,p}(\overline{u}_{l+p}) & \cdots & N_{k,p}(\overline{u}_{l+p})\end{bmatrix}$$

These two sets of equations completely determine the Lagrangian weights $\{\varpi_i\}_{i=l}^{l+p}$ and the weighted nodes $\boldsymbol{P}_i^{\varpi}$. The values of the physical nodes follow the following form $\boldsymbol{P}_i=\boldsymbol{P}_i^{\varpi}/\varpi_i,\ i=l,\dots,l+p$. Note that, by construction, this transformation preserves the original geometry.

It should be emphasized that this transformation is defined at the element level. It depends only on the knot vector and the placement of Lagrangian knots, not the control points. The transformation for each element can be pre-computed and stored. The transformation is invertible, and the inversion gives a linear transformation between the basis functions:

$$\begin{bmatrix}L_{p,l}\\ L_{p,l+1}\\ \vdots\\ L_{p,l+p}\end{bmatrix}=\mathbf{M}^{-1}\begin{bmatrix}N_{k-p,p}\\ N_{k-p+1,p}\\ \vdots\\ N_{k,p}\end{bmatrix} \tag{6}$$

This relation allows us express the Lagrangian basis in terms of the B-spline basis over a segment. The relation will be used later.

2.2. NURBS surface

The transformation to NURBS surface proceeds in the same way. Let us consider an element in a $p\times q$ degree NURBS mesh. Let $\{Q_{ij}\}$ and $\{\omega_{ij}\}$, $i=1,p+1;j=1,q+1$, be the element control points and weights, respectively. The element is represented as

$$\boldsymbol{S}(u,v)=\frac{\sum_{i=1}^{p+1}\sum_{j=1}^{q+1}N_{i,p}(u)N_{j,q}(v)\omega_{ij}\mathbf{Q}_{ij}}{\sum_{i=1}^{p+1}\sum_{j=1}^{q+1}N_{i,p}(u)N_{j,q}(v)\omega_{ij}} \tag{7}$$

We seek to represent it as

$$\boldsymbol{S}(u,v)=\frac{\sum_{i=1}^{p+1}\sum_{j=1}^{q+1}L_{i,p}(u)L_{j,q}(v)\varpi_{ij}\boldsymbol{P}_{ij}}{\sum_{i=1}^{p+1}\sum_{j=1}^{q+1}L_{i,p}(u)L_{j,q}(v)\varpi_{ij}} \tag{8}$$

where $L_{i,p}(u)$ and $L_{j,q}(v)$ are Lagrangian polynomials defined in the u- and v-parametric coordinates. The Lagrangian nodes in the parametric coordinates are taken to be uniformly placed, $\overline{u}_{1+i}=$