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Violin string shape functions for finite element analysis of rotating Timoshenko beams

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ABSTRACT

Violin strings are relatively short and stiff and are well modeled by Timoshenko beam theory. We use the static part of the homogeneous differential equation of violin strings to obtain new shape functions for the finite element analysis of rotating Timoshenko beams. For deriving the shape functions, the rotating beam is considered as a sequence of violin strings. The violin string shape functions depend on rotation speed and element position along the beam length and account for centrifugal stiffening effects as well as rotary inertia and shear deformation on dynamic characteristics of rotating Timoshenko beams. Numerical results show that the violin string basis functions perform much better than the conventional polynomials at high rotation speeds and are thus useful for turbo machine applications.

1. Introduction

Rotating cantilever beams are used in propellers, turbines, wind turbines and helicopter rotor blades, etc. [1,2]. Gas and steam turbine blades are short and rigid and can be modeled as rotating Timoshenko beams [3]. Rotating Euler–Bernoulli beams only consider centrifugal force in addition to inertial and elastic forces for vibration analysis. The secondary effects such as shear deformation and rotary inertia have a small effect on lower modes but have considerable effect on higher modes [3]. Hence for accurate prediction of higher modes, Timoshenko beam models are employed.

Vibration analysis of non-rotating Timoshenko beams has been addressed by a some investigators using the finite element method. Many of these studies experienced difficulty in incorporating all the boundary conditions. Kapur [4] used a finite element model approach for finding the frequencies and mode shapes for free vibrations of a uniform or nonuniform Timoshenko beam for various boundary conditions. Thomas and Abbas [5] analyzed a uniform Timoshenko beam by taking the total deflection, the total slope, bending slope and derivative of bending slope as internal degrees of freedom. The frequency equation for Timoshenko beams was developed by van Rensburg and van der Merve [6] and properties of natural frequencies and mode shapes such as double eigenvalues, significance of dimensionless parameters and estimates of large and small eigenvalues were studied for various boundary conditions. Bokanian [7] developed exact frequency equations for axially loaded beams. Özdemir and Kaya [8] performed vibration analysis of Timoshenko beams using the differential transform method. Friedman and Kosmatka [9] developed a two node finite element for a Timoshenko beam which satisfied the static homogeneous governing differential equations. Kosmatka [10] also developed a two node finite element for an axially loaded Timoshenko beam. Although some of the methods presented in cited references take into account axial loads, they are restricted to non-rotating tapered beams.

The dynamics of rotating Timoshenko beams differs from that of non-rotating Timoshenko beams since the centrifugal forces effect the computed frequencies and mode shapes in addition to shear deformation and rotary inertia. Also, the partial differential equation for rotating beams has variable coefficients compared to that of non-rotating beam which has constant coefficients. Therefore, the rotating beam equation is not analytically solvable even for Euler–Bernoulli beams and various approximate methods have been used in the literature. Hodges and Rutkowski [11] developed a variable order finite element model with displacement functions as power series. Frobenius method of series solution was used by Naguleswaran [12]. Banerjee [13] used the dynamic stiffness method for free vibration of centrifugally stiffened uniform and tapered beams where frequency dependant shape functions are obtained from governing differential equation. Özdemir and Kaya [14] used the differential transform

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method, a semi analytical-numerical technique to compute natural frequencies. It is based on Taylor series expansion where both the governing differential equations of motion and the boundary conditions of the system were transformed into a set of algebraic equations using certain transformation rules. Wang and Wereley [15] and Vinod et al. [16] used spectral finite element (SFEM) to study free vibration and wave propagation characteristics of rotating Euler–Bernoulli beams. The dynamic stiffness matrix was constructed using the weak form of the governing differential equation in the frequency domain and resulting lower order eigenvalue problem was solved. They discussed drawbacks of available finite element methods which resulted in large size eigenvalue problem. Recent works which include Fourier-p finite element [17] advocate such lower order models which are used in vibration control applications and allow for easy inclusion of non-uniformities for flexural stiffness and mass distribution.

Accurate approaches to develop finite elements involve selecting shape functions such as stiff string functions [18], hybrid stiff string functions [19], higher order polynomials [20], trigonometric functions [21], rational interpolations functions [22] and beam displacement function [23]. Some of these basis functions try to satisfy the *static* part of governing differential equations of the Euler–Bernoulli rotating beam. The stiff string equation, for instance, is used to model piano strings and can be obtained by adding a flexural stiffness term to the wave equation governing string vibration.

However, Timoshenko beams are better models for many rotating beam structures. Yokohama [24] studied free vibration characteristics of rotating Timoshenko beams addressing the hub radius and setting angle effects. Lin et al. [25] studied the instability and vibration of a rotating Timoshenko beam with precone. Banerjee and Sobey [26] gave energy expressions for rotating tapered Timoshenko beams. Banerjee [27] extended the dynamic element formulation of a rotating Euler–Bernoulli beam to account for secondary effects. Du et al. [28] solved the partial differential equation of a rotating Timoshenko beam using the power series method. Kaya [29] extended their works for rotating uniform and tapered Timoshenko beams using DTM. Rao and Gupta [30] used finite element approach to study effects of twist, offset, speed of rotation and variation of breadth and taper ratios on natural frequencies and mode shapes. Bazoune and Khulief [31] used finite element for vibration analysis of double tapered Timoshenko beams for various boundary conditions. Bambill et al. [32] used differential quadrature method for vibration analysis of rotating Timoshenko beams. In this method the derivatives are approximated using the weighted linear summation of functional values at all sample points in overall domain. Using this approximation the differential equation is transformed into set of algebraic equations. The number of equations depends on number of sample points taken in the domain.

In this paper, we seek to find better shape functions for rotating Timoshenko beams using the static part of governing differential equation of a violin string. Maezawa et al. [33] have shown that Timoshenko beam theory gives a better estimate of the inharmonicity effects in violin strings compared to Euler–Bernoulli beam theory. Inharmonicity in violin strings occurs because flexural stiffness and other effects which are present in real strings cause the frequency of the nth mode to deviate from the exact nth multiple of the fundamental natural frequency. Interestingly, we found that the violin string equations can be obtained by assuming *constant tension* in the rotating beam equation. Moreover, the violin string static homogenous differential equations have an *exact solution*. As the violin string is the closest approximation to rotating Timoshenko beams for which an exact solution exists, shape functions derived from them can exhibit better convergence compared to cubic elements for predicting the frequencies.

2. Timoshenko rotating beam and violin string

The governing equation for free vibrations of a rotating Timoshenko beam is given by [10]

$$\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2}-\frac{\partial}{\partial x}\left(T(x)\frac{\partial w(x,t)}{\partial x}\right)-\frac{\partial}{\partial x}\left(GA(x)k\left(\frac{\partial w(x,t)}{\partial x}-\theta(x,t)\right)\right)=0 \tag{1}$$

$$\rho I(x)\frac{\partial^2 \theta(x,t)}{\partial t^2}-GA(x)k\left(\frac{\partial w(x,t)}{\partial x}-\theta(x,t)\right)-\frac{\partial}{\partial x}\left(EI(x)\frac{\partial \theta(x,t)}{\partial x}\right)=0 \tag{2}$$

where $T(x)$ is axial force due to centrifugal stiffening and is given by

$$T(x)=\int_x^L \rho A(x)\Omega^2\, dx \tag{3}$$

Here L is the length of beam, Ω is the rotational speed, $\theta(x,t)$ is the angle of rotation of cross-section, $w(x,t)$ is the vertical displacement of beam, ρ is the density, E and G are elastic constants, k is the shear coefficient, $A(x)$ is area of cross-section, $I(x)$ is moment of inertia of cross-section as shown in Fig. 1. The non-dimensional constants are $\nu=AL^2/I$, $\beta=GAkL^2/EI$ and $\gamma=\beta/\nu=Gk/E$, $\kappa^2=\rho A\Omega^2 L^4/EI$. For beams with constant tension, Eqs. (1) and (2) reduce to

$$\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2}-T\frac{\partial^2 w(x,t)}{\partial x^2}-\frac{\partial}{\partial x}\left(GA(x)k\left(\frac{\partial w(x,t)}{\partial x}-\theta(x,t)\right)\right)=0 \tag{4}$$

$$\rho I(x)\frac{\partial^2 \theta(x,t)}{\partial t^2}-GA(x)k\left(\frac{\partial w(x,t)}{\partial x}-\theta(x,t)\right)-\frac{\partial}{\partial x}\left(EI(x)\frac{\partial \theta(x,t)}{\partial x}\right)=0 \tag{5}$$

where T is the constant axial tension. We call these the violin string equations in an analogy to the typical Euler–Bernoulli stiff string equations [17]. The stiff strings are studied in the acoustic analysis of piano strings. Piano strings have flexural stiffness but are typically quite long. In fact, the most prized pianos are those with the largest strings. Violins are much smaller than pianos. Timoshenko theory gives a better representation of inharmonicity of violin strings compared to Euler–Bernoulli beam theory [33]. The static part of Eqs. (4) and (5) is obtained by ignoring the inertial term which leads to

$$T\frac{d^2 w}{dx^2}+\frac{d}{dx}\left(GA(x)k\left(\frac{dw}{dx}-\theta\right)\right)=0 \tag{6}$$

$$GA(x)k\left(\frac{dw}{dx}-\theta\right)+\frac{d}{dx}\left(EI(x)\frac{d\theta}{dx}\right)=0 \tag{7}$$

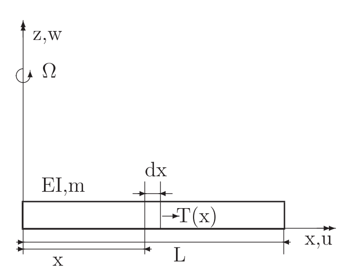

Fig. 1. Rotating beam.

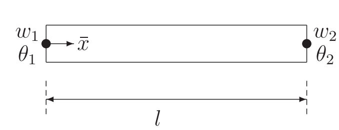

Fig. 2. Beam element.