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Efficient coupled polynomial interpolation scheme for out-of-plane free vibration analysis of curved beams

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ABSTRACT

The performance of two curved beam finite element models based on coupled polynomial displacement fields is investigated for out-of-plane vibration of arches. These two-noded beam models employ curvilinear strain definitions and have three degrees of freedom per node namely, out-of-plane translation (v), out-of-plane bending rotation (θ_z) and torsion rotation (θ_s). The coupled polynomial interpolation fields are derived independently for Timoshenko and Euler–Bernoulli beam elements using the force-moment equilibrium equations. Numerical performance of these elements for constrained and unconstrained arches is compared with the conventional curved beam models which are based on independent polynomial fields. The formulation is shown to be free from any spurious constraints in the limit of 'flexureless torsion' and 'torsionless flexure' and hence devoid of flexure and torsion locking. The resulting stiffness and consistent mass matrices generated from the coupled displacement models show excellent convergence of natural frequencies in locking regimes. The accuracy of the shear flexibility added to the elements is also demonstrated. The coupled polynomial models are shown to perform consistently over a wide range of flexure-to-shear (EI/GA) and flexure-to-torsion (EI/GJ) stiffness ratios and are inherently devoid of *flexure*, *torsion* and *shear locking* phenomena.

1. Introduction

The out-of-plane free vibration of curved beams exhibit coupled flexure and torsion behavior. The presence of the shear, flexure and torsion locking phenomena in the out-of-plane deformation of conventional curved beam finite element models [1,2] significantly influence their dynamic behavior. The conventional curved Timoshenko beam finite element model exhibit flexure and torsion locking in addition to shear locking, whereas Euler–Bernoulli beam model exhibit only torsion locking [2]. The severity of the flexure and torsion locking phenomena was shown to depend on the magnitude of flexure-to-torsion (EI/GJ) stiffness ratio. It was demonstrated that, the use of inconsistent torsion strain models produces spurious torsion strain energy and fail to simulate accurately the 'torsionless flexure' behavior when $GJ \gg EI$, leading to torsion locking. Similarly, the use of inconsistent flexure strain models produce spurious flexure strain energy and fail to simulate accurately the 'flexureless torsion' behavior when $EI \gg GJ$, leading to flexure locking.

Major research on the out-of-plane vibration analysis of curved beams mainly focused on developing models to accurately represent the dynamic characteristics [3–15]. Other studies focused on understanding the effect of shear deformation and rotary inertia on the frequency and mode shapes of thick and thin beams under constrained and unconstrained end conditions [16–19]. Recently, the authors [1,2], investigated flexure and torsion locking phenomena in out-of-plane deformation of Timoshenko and Euler–Bernoulli curved beams. Improved curved beam finite elements based on coupled displacement field methodology were proposed to alleviate these locking effects and to enhance the convergence rate. In the above papers, the authors have evaluated/examined performance of these proposed elements in the context of static analysis. However, the use of coupled shape functions derived based on the static equilibrium consideration, for computation of consistent element mass matrix for dynamic analysis is questionable and the performance of such a finite element is not obvious. The focus of the current paper is to verify and validate the applicability of the above coupled shape functions to dynamic analysis. The important aspect of the shape functions is that their coefficients are coupled, which however, do not introduce/produce

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any spurious modes even in the extreme limits of stiffness ratios. The efficacy of the consistent mass matrix is examined and ascertained in predicting the natural frequencies of curved arches under different boundary conditions. Also, the performance of the coupled models is compared with the conventional finite element models which are based on independent field interpolations. Further, the applicability and performance is verified in the shear, flexure and torsion locking regimes.

This article is organized as follows: In Section 2, the geometry and coordinate systems for a curved beam element are described and the finite element formulation based on Hamilton's principle is presented. In Section 3, the independent polynomial fields for conventional models are presented and the constraints are examined in the context of flexure and torsion locking. In Section 4, the displacement fields for coupled polynomial models are presented and the role of the coupled terms in eliminating locking is examined in detail. In Section 5, numerical studies are carried out to demonstrate the efficacy of the coupled models in predicting natural frequencies of curved beam structures under different end constraints. These models exhibit consistent performance over a wide range of flexure-to-shear (EI/GA) and flexure-to-torsion (EI/GJ) stiffness ratios and are inherently devoid of shear, flexure and torsion locking phenomena. Conclusions are presented in Section 6.

2. Finite element formulation

The geometry and coordinate system for a curved beam element of length $2L$ and radius of curvature R is shown in Fig. 1. A right-handed orthogonal curvilinear co-ordinate system s-y-z is used with its origin placed at the center of the element. The natural coordinate ξ along the circumferential direction is expressed as $\xi = s/L$.

The out-of-plane flexure, shear and torsion strain components in the curvilinear co-ordinate system are

$$\kappa = -\theta_z' + \frac{\theta_s}{R} \tag{1}$$

$$\gamma = v' - \theta_z \tag{2}$$

$$\tau = \theta_s' + \frac{\theta_z}{R} \tag{3}$$

In the above equations, v is the transverse out-of-plane displacement, θ_z is the flexure rotation and θ_s is the torsional rotation of the cross-section. The prime ($'$) indicates the derivative with respect to the circumferential coordinate s.

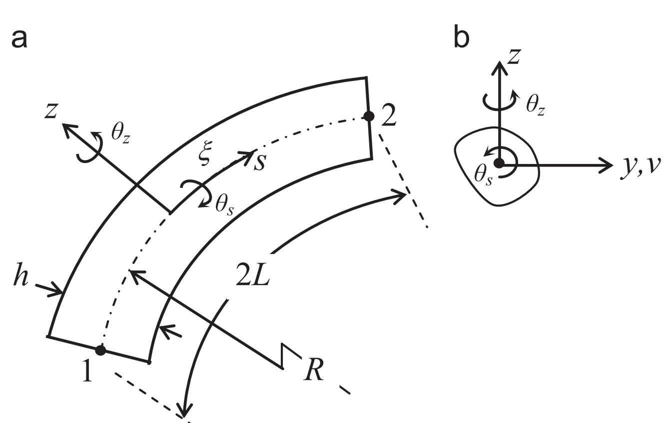

Fig. 1. Coordinate system for the 2-noded curved beam element: (a) geometry; (b) cross-section.

The strain energy stored in the element is the sum of energies due to shear, flexure and torsion deformation and is written as

$$U_e(v,\theta_z,\theta_s) = \frac{1}{2}\int_{-L}^{L}\left[kGA\ (v'-\theta_z)^2 + EI_z\left(-\theta_z' + \frac{\theta_s}{R}\right)^2 + GJ\ \left(\theta_s' + \frac{\theta_z}{R}\right)^2\right]ds \tag{4}$$

The terms in the square bracket represent the strain energy due to transverse shear deformation, bending and torsion respectively; E and G denote Young's modulus and shear modulus, respectively, k is the shear correction factor; and R is the radius of curvature of the beam element. The geometric parameters A, I_z and J denote the area, moment of inertia, and torsional constant for the cross-section, respectively.

The kinetic energy stored in the element consists of kinetic energy due to translation motion and rotary inertia due to bending and torsion is expressed as

$$T_e(v,\theta_z,\theta_s) = \frac{1}{2}\int_{-L}^{L}\left[\rho A\dot{v}^2 + \rho I_z\dot{\theta}_z^2 + \rho I_s\dot{\theta}_s^2\right]ds \tag{5}$$

Neglecting the work done by applied forces, the equation of motion is obtained by applying Hamilton's principle [20] as

$$0 = \int_{t_1}^{t_2} \delta(T-U)\,dt \tag{6}$$

Performing integration by parts we obtain the equation of motion as

$$\mathbf{M}\,\ddot{q} + \mathbf{K}\,q = 0 \tag{7}$$

where $q = \{v\ \ \theta_z\ \ \theta_s\}^T$.

Assuming polynomial expressions for the field variables v, θ_z and θ_s as

$$v = \sum_{i=1}^{l} v_i\ \phi_i, \quad \theta_z = \sum_{i=1}^{m} \theta_{zi}\ \psi_i, \quad \theta_s = \sum_{i=1}^{n} \theta_{si}\ \chi_i \tag{8}$$

and substituting in Eq. (7) we obtain the expressions for stiffness '$\mathbf{K}$' and mass '$\mathbf{M}$' as

$$\mathbf{K}_{ij}^{11} = \int_{-L}^{L}\left[GAk_s\frac{d\phi_i}{ds}\frac{d\phi_j}{ds}\right]ds, \quad \mathbf{K}_{ij}^{12} = -\int_{-L}^{L}\left[GAk_s\frac{d\phi_i}{ds}\psi_j\right]ds = K_{ji}^{21}$$

$$\mathbf{K}_{ij}^{22} = \int_{-L}^{L}\left[G\,Ak_s\psi_j\psi_i + GJ\frac{\psi_j}{R}\frac{\psi_i}{R} + EI\frac{d\psi_i}{ds}\frac{d\psi_i}{ds}\right]ds$$

$$\mathbf{K}_{ij}^{23} = \int_{-L}^{L}\left[GJ\frac{\psi_j}{R}\frac{d\chi_i}{ds} + EI\frac{d\psi_i}{ds}\frac{\chi_i}{R}\right]ds = \mathbf{K}_{ji}^{32},$$

$$\mathbf{K}_{ij}^{33} = \int_{-L}^{L}\left[GJ\frac{d\chi_j}{ds}\frac{d\chi_i}{ds} - EI\frac{\chi_j}{R}\frac{\chi_i}{R}\right]ds,$$

$$\mathbf{K}_{ij}^{13} = \mathbf{K}_{ji}^{31} = 0$$

$$\mathbf{M}_{ij}^{11} = \int_{-L}^{L}\left[\rho A\ \phi_j\phi_i\right]ds, \quad \mathbf{M}_{ij}^{22} = \int_{-L}^{L}\left[\rho I_z\psi_j\psi_i\right]ds,$$

$$\mathbf{M}_{ij}^{33} = \int_{-L}^{L}\left[\rho I_s\chi_j\chi_i\right]ds$$

The elemental Eq. (7) are assembled to obtain global equations of motion. These second order ordinary differential equations are converted into an Eigenvalue problem by assuming harmonic motion as

$$\left[\mathbf{K} - \omega^2\mathbf{M}\right]\ q = 0 \tag{9}$$

where ω are the eigenvalues and q are eigenvectors.

The accuracy and convergence characteristic of an element depends on the choice of the polynomial assumed for the field variables. Therefore, the selection of a proper displacement field for the element is of utmost important. In the following sections, for a two-noded curved beam element, the displacement fields for independent polynomial models and coupled polynomial models