Finite Elements in Analysis and Design 47 (2011) 1253–1261

Contents lists available at ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.com/locate/finel

A robust continuation method to pass limit-point instability

Kavous Jorabchi, Krishnan Suresh *

Mechanical Engineering Department, University of Wisconsin–Madison, 2050 Mechanical Engineering Building, 1513 University Ave., Madison, WI, USA

ARTICLE INFO

Article history:
Received 23 January 2011
Received in revised form
18 May 2011
Accepted 19 May 2011
Available online 2 July 2011

Keywords:
Geometric nonlinearity
Finite element analysis
Structural stability
Snap-through
Limit points
Ill-conditioning

ABSTRACT

In this paper a homotopy map is proposed to pass limit points of snap-through problems encountered in geometrically nonlinear finite element analysis. In the vicinity of such points, the tangent stiffness matrix becomes ill-conditioned, which detrimentally affects the convergence of numerical schemes such as Newton–Raphson method.

The proposed method overcomes this problem by tracing a well-conditioned path instead of equilibrium path in the vicinity of critical points. This allows the solution procedure to bypass the critical point without experiencing ill-conditioning. An instance of such a well-conditioned path is constructed for limit points. In particular, starting from the stable (or unstable) configuration, we compute the unstable (or stable) configuration via a robust numerical procedure. Further, since the tangent matrix derivation is consistent with the residual force computation, the quadratic convergence of Newton–Raphson method is retained.

1. Introduction

Stability analysis is one of the most important design considerations in structural engineering. Many structures such as bars, beams, plates and shells (which have at least one dimension much smaller than others) can exhibit structural instability under certain loading conditions even when the loads are well below yield point of constituent material. Such behavior is not associated with material failure but rather a significant configurational change in structure. Hence, the problem of elastic instability inevitably requires use of nonlinear theory of elasticity where one needs to account for geometric nonlinearities and large deformations.

Stability analysis of geometrically nonlinear elastic structures entails obtaining the entire load–displacement path. However, computing the load–displacement path can be challenging due to existence of critical points. Critical points are commonly categorized into bifurcation points and limit points [1] as shown in Fig. 1. This figure also illustrates another class of points known as turning points. Turning points are regular points and have less physical/computational significance [2]. The focus of this paper is on limit points.

In the vicinity of a limit point, the tangent stiffness matrix of finite element formulation becomes ill-conditioned giving rise to two problems: (1) the underlying algebraic system of equations becomes harder to solve using numerical solvers [3,4], (2) solution jumps to a distant stable configuration making it harder for a numerical method to converge [5]. Numerous techniques, reviewed below, have been proposed to overcome these two problems. We only cover the techniques that are concerned with geometrically nonlinear Finite Element Analysis (FEA).

Bergan [6] proposed to suppress equilibrium iterations until the limit point is passed. This solves both problems; however, the technique unfavorably produces a drift from equilibrium path. An alternative technique was proposed by Wright and Gaylord [7] that entails adding a fictitious spring to stabilize the tangent stiffness matrix in the vicinity of a limit point. However, their approach appears to be unsuitable for general structures.

Argyris [8] proposed a class of methods referred to as displacement control methods. Different variations of these methods are formulated for example in [9,10]. The method in [10], for instance, preserves symmetry and banded form of tangent stiffness matrix. Displacement control methods successfully overcome abovementioned problems. However, they fail to trace the equilibrium path beyond a turning point. Moreover these methods implicitly assume that there exists at least one degree of freedom with a monotonic evolution. However, such a degree of freedom may not exist (see for example [11]), and even if it exists, there is no systematic approach to find it.

Thurston et al. [12] proposed a different technique where modal transformation was used to control the behavior of ill-conditioned modes associated with small eigenvalues. However, this method requires computation of higher order terms in residual vector in order to make the resulting modal equations consistent; hence, the method is computationally expensive.

* Corresponding author. Tel.: +608 262 3594; fax: +608 265 2316.
E-mail addresses: kjorabchi@wisc.edu (K. Jorabchi), suresh@engr.wisc.edu (K. Suresh).

0168-874X/$ - see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.finel.2011.05.012

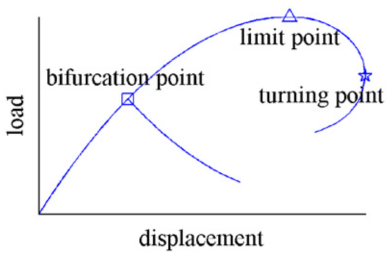

Fig. 1. Typical load–displacement graph.

Clarke and Hancock [13] summarized yet another class of methods which are obtained by augmenting FEA equations with a constraint equation. Depending upon the type of the constraint equation, many techniques have been derived among which arc-length methods [1,5,14–17] have gained popularity over the past years. Further developments in arc-length type methods are summarized in [18–24].

Arc-length methods are well-established and have been widely used in commercial finite element packages. However, as Müller [3,4] mentioned, these methods suffer from ill-conditioning in the vicinity of critical points in that "*numerical defect of the stiffness matrix is usually not repaired (exception: Wriggers and Simo* [25], *Felippa* [26]*). It is commonly assumed that during iteration the critical point is not precisely hit*". In case of a precise hit, the solution is usually perturbed and the load step is repeated [27]. Riks [16] showed that this shortcoming stems from particular formulation of constraint equation. An alternative formulation was proposed in [16] that led to a robust algorithm near limit points. However, this technique does not generalize to all constraint equations. Moreover, one needs to employ *linearized* constraint equation at each corrector step (unlike Crisfield's method [5]). Crisfield et al. [28] reported severe difficulties with conventional cylindrical arc-length method and appealed to hybrid static/dynamic procedure to overcome these difficulties. Further failure modes of arc-length methods are summarized by Carrera in [29].

For the reasons mentioned above, Belytschko et al. [30] believe that "*tracing of equilibrium branches is often quite difficult; robust and automatic procedures for continuation are not yet available*". To address these challenges, Müller [3,4] proposed a stabilized Newton–Raphson method. Stabilization methods are widely used in commercial FEA packages. However, we identify following shortcomings with such techniques:

1) Larger number of iterations might be required to jump between two successive, but far apart, stable configurations.
2) Quadratic convergence of Newton method is compromised due to inconsistency between the stabilized tangent matrix and residual vector.
3) Only the *loading* path is captured as shown in Fig. 2. As can be observed in this figure, there exists a stable portion of equilibrium path which is *not* traced during loading, however, this portion will be traced during unloading. Although stabilization methods can be modified to compute the unloading path, this will require additional iterations.
4) The topology of the equilibrium path may not be preserved. In other words, stable but disconnected equilibrium paths may merge giving the analyst a wrong conclusion about structure's response in practice. Such paths are frequently observed for imperfect systems; see for example [31].

For these reasons, we believe that there are *computational* merits to trace the entire equilibrium path, despite the fact that only stable branches of a system have *practical* significance.

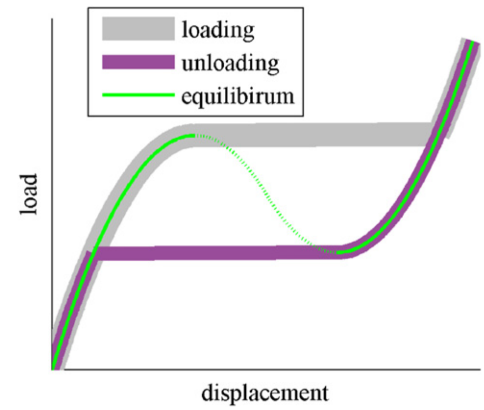

Fig. 2. Loading and unloading paths.

The proposed method in this paper relies on the concept of homotopy [32] (also referred to as continuation) to overcome the abovementioned problems. The main concept behind homotopy methods is as follows: first an "easy" system of equations to which the solution is trivially obtained is set up; this easy system is then gradually transformed into the original system of nonlinear equations via a control parameter. Homotopy methods have received considerable attention for solving nonlinear differential and algebraic equations, see for example [33,34] and references therein. More recently, these methods have been successfully applied to solve different instability problems. For examples, Fujii and Okazawa [35] used homotopy path in conjunction with local iterations to compute the stability points of structures. Researchers in [36] solved pull-in instability problem of electromechanical systems via homotopy method. A higher order iterative-corrector method based on homotopy transformation was proposed in [37] and applied to geometrically nonlinear problems.

In this paper, we exploit the homotopy concept to arrive at a robust Newton–Raphson technique. In particular, we construct a different (and well-conditioned) path instead of equilibrium path in the vicinity of critical points to bypass these points. An instance of such a path is derived for limit points in Section 3. Through an adaptive framework, we ensure that the tangent matrix along the path is well-conditioned. Consequently, the proposed technique finds the unstable (or stable) configuration of the system from stable (or unstable) configuration for the fixed load level, essentially *jumping* over the limit point.

The remainder of the paper is organized as follows. We set up general FEA equations in the context of large deformation elasticity in Section 2. The proposed method is formally established in Section 3. Adaptive selection of stabilization parameters is discussed in Section 4. Section 5 presents several numerical examples, followed by conclusion and future work in Section 6.

2. General FEA equations

Recall that finite element discretization of large displacement elasticity problems results in a system of nonlinear algebraic equations of the form [38]

$$\boldsymbol{\Psi}(\mathbf{u},\lambda) = \mathbf{F}_{\text{int}}(\mathbf{u}) - \lambda\mathbf{F}_{\text{ext}} = \mathbf{0} \tag{2.1}$$

where $\boldsymbol{\Psi}$ is the residual vector, $\mathbf{F}_{\text{int}}(\mathbf{u})$ is the internal force vector which is a nonlinear function of displacement $\mathbf{u}$, and $\mathbf{F}_{\text{ext}}$ is the normalized external load vector which is assumed to be independent of $\mathbf{u}$. The magnitude of the external load is controlled by λ that is varied as the equilibrium path is traced. The standard