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# An augmented Lagrangian contact formulation for frictional discontinuities with the extended finite element method

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ABSTRACT

In this paper, an *Uzawa*-type augmented Lagrangian contact formulation is presented for modeling frictional discontinuities in the framework of the X-FEM technique. The kinematically nonlinear contact problem is resolved based on an *active set strategy* to fulfill the Kuhn–Tucker inequalities in the normal direction of contact. The Coulomb's friction rule is employed to address the stick–slip behavior on the contact interface through a return mapping algorithm in conjunction with a *symmetrized* (nested) augmented Lagrangian approach. A stabilization algorithm is proposed for the robust imposition of the frictional contact constraints within the proposed augmented Lagrangian framework. Several numerical examples are presented to demonstrate various aspects of the proposed computational algorithm in simulation of the straight, curved and wave-shaped discontinuities.



## 1. Introduction

Numerical simulation of frictional contact behavior between two bodies has enormous applications in broad range of engineering areas. For instance, in the forming process relative movement between the tools and material produces normal and tangential contact forces, which have an important effect on various aspects of the process, such as the pressing forces, density distribution, crack nucleation, and residual stresses [1,2]. Moreover, in geomechanical problems the crack growth in compressive zone, such as fault rapture phenomenon, involves frictional sliding of the crack edges [3,4]. The shearing forces on crack faces are determinant in the computation of stress intensity factors, rate of propagation, and direction of propagation; and as a result can drastically affect the behavior of cracked media. Modeling of frictional contact in continuum mechanics produces complexity within the solution, in which two nested Kuhn–Tucker constraints should be satisfied, including: an outer constraint that accounts for the contact/separation mode, and an inner constraint that describes the stick/slip condition on contacting boundaries.

In cases that the contact faces are well defined prior to beginning of the simulation, the finite element method provides a natural means for simulating frictional and frictionless contact problems, see e.g. [5–7]. However, there exist a large variety of applications where the contacting boundaries nucleate in an initially undamaged medium and evolve in an undetermined direction through the simulation, such as mode II and III crack propagation problems. The eXtended Finite Element Method pioneered by Moës et al. [8] circumvents the need for conforming the FE mesh to the evolving boundaries of the discontinuity by enriching the standard FE approximation with additional discontinuous interpolation functions on the basis of the partition of unity method [9,10]. The X-FEM requires no update in the mesh topology, and the only interaction between the FE mesh and the discontinuity involves selection of the nodal points that must be enriched [11,12]. A comprehensive review on X-FEM and its application in a variety of problems in continuum mechanics can be found in [13,14]. In particular, both primal and dual formulations have been pursued to model frictional contact within the context of X-FEM. As a pioneering work, Dolbow et al. [15] employed the LArge Time INcrement (LATIN) method to model frictional contact on embedded interfaces, in which the solution is decomposed into two global (linear) and local (possibly nonlinear) steps, see also [16]. The conventional penalty method has also been employed to model frictional contact problems with the X-FEM in the works of Khoei and Nikbakht [17] and Liu and Borja [18], and is then extended to large sliding-large deformation contact problems [19,20].

It is well recognized that irrespective of whether one employs the penalty method or the Lagrangian approach, imposing the Dirichlet (or stiff Neumann) constraints on embedded interfaces, such as those arising in X-FEM modeling of frictional contact, may cause numerical instability in forms of spurious oscillation in the interfacial traction/flux fields and loss of convergence in local error

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norms [21–23]. Typically, these oscillations are more severe in the Lagrange multipliers method where contact constraints are imposed exactly, however, the penalty method exhibits the same effect when a high-stiff penalty number is employed to impose the constraints more accurately. In these circumstances, the key challenge is the verification of the so-called Ladyzhenskaya–Babuška–Brezzi condition [24,25], which ensures a proper choice of the discrete space for the Lagrange multipliers in order to avoid an over-constrained solution. Basically, the issue of verifying the LBB (or inf-sup) condition is related to the non-conformity of the X-FEM mesh to the Dirichlet boundaries and stems from the same challenges encountered when dealing with non-conforming Dirichlet boundaries in standard finite element frameworks [26,27]. Ji and Dolbow [21] emphasized that the most convenient choice of basis for the discrete Lagrange multipliers, those that naturally arise from the intersections of the interface with the bulk mesh, triggers oscillation in the Lagrange multipliers field and cause lack of convergence in local error norms.

So far, both stable and stabilized approaches have been adapted to the X-FEM models of contact problems to restore stability of contact stress fields. In general, the discrete Lagrange multipliers space needs to be coarsened with respect to the underlying mesh to recover stable Lagrange multipliers on embedded interfaces. This can be achieved by constructing a stable discrete Lagrange multipliers space based on the vital vertex algorithm proposed by Moës et al. [22] and then improved by Bechet et al. [28] and Hautefeuille et al. [29], or through adopting a mortar-like technique which employs an independent mesh for the discontinuity as proposed by McDevitt and Laursen [30] and Kim et al. [31]. The former approach is also extended for large sliding contact by Nistor et al. [32] and Siavelis et al. [33]. Furthermore, a stable discrete space can be achieved by an additional enrichment of the displacement basis with bubble function as given in [23,34]. On the other hand, the stabilized techniques mostly rely on the Nitsche's approach adopted to model frictional contact on embedded interfaces in [35–37]. It was shown by Stenberg [38] that the bubble stabilization method is closely related to Nitsche's approach. There are also a number of stabilized global–local approaches which allow an independent discretization of the bulk and the crack interface, developed in conjunction with the LATIN strategy in non-planar frictional crack and fatigue crack growth [39–41]. A stabilized lower order formulation for frictional contact problems based on the polynomial pressure projection (PPP) method was proposed by Liu and Borja [42] that is applicable to both penalty and Lagrange multipliers methods.

In the present study, an augmented Lagrangian formulation is presented to model frictional discontinuities on the basis of an extended finite element method. The main idea is to combine the penalty and Lagrange multipliers methods to inherit the advantages of both approaches, in order to decrease the ill-conditioning of governing equations and to satisfy the contact constraints more accurately with finite values of penalty parameters. The proposed variational formulation is used to incorporate the frictional contact behavior into the formulation. The finite element equations consist of two sets of equations; a standard FE equation without the discontinuity and an enhanced FE equation that captures the discontinuity and contact contribution via the enriched degrees of freedom. An active set strategy is employed to resolve the kinematically nonlinear equations in conjunction with the Newton–Raphson iterative solution strategy. In order to address the frictional behavior at the contact interface, the plasticity theory of friction is employed based on the Coulomb's friction law, which is implemented via the standard return mapping algorithm on the basis of a symmetrized (nested) augmented Lagrangian approach. Finally, several numerical examples are simulated to demonstrate various aspects of proposed computational algorithm, including the combined opening–closing and sticking–slipping conditions in frictional straight, curved, and wave-shaped discontinuities.

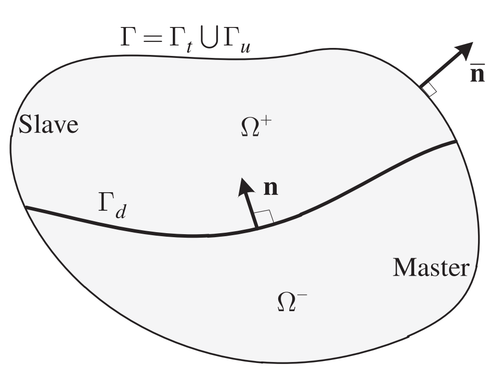


**Fig. 1.** Definition of a contact problem; The discontinuity $\Gamma_d$ separates the domain $\Omega$ into the slave ($\Omega^+$) and master ($\Omega^-$) bodies.

## 2. Governing equations of contact problem

Consider a two-dimensional body $\Omega$ bounded externally by $\Gamma$, in which the discontinuity $\Gamma_d$ separates the body into $\Omega^+$ and $\Omega^-$, as illustrated in Fig. 1. The boundary value problem governing the static equilibrium equation of the body can be written as

$$\begin{aligned} &\nabla.\boldsymbol{\sigma}+\mathbf{b}=\mathbf{0} && \text{in} && \Omega, \\ &\mathbf{u}=\tilde{\mathbf{u}} && \text{on} && \Gamma_u \subset \Gamma, \\ &\boldsymbol{\sigma}\,\bar{n}=\tilde{\mathbf{t}} && \text{on} && \Gamma_t \subset \Gamma, \end{aligned} \tag{1}$$

where $\nabla$ is the gradient operator, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{b}$ is the body force vector, $\tilde{\mathbf{u}}$ is the prescribed displacement on $\Gamma_u$, and $\tilde{\mathbf{t}}$ is the prescribed traction acting on $\Gamma_t$ whose unit outward normal vector is denoted by $\bar{\mathbf{n}}$. It is assumed that $\Gamma_u \cup \Gamma_t = \Gamma$ and $\Gamma_u \cap \Gamma_t = \varnothing$.

The notion of '*slave*' and '*master*' bodies of the classical contact mechanics can also be applied here, such that the master and slave bodies in the current case pertain to each side of the discontinuity $\Gamma_d$, where $\Omega^+$ is assigned to the slave side and $\Omega^-$ to the master side. On account of the frictional contact condition along the discontinuity interface $\Gamma_d$, the above mentioned boundary value problem is augmented by the following condition on the surface of discontinuity

$$\boldsymbol{\sigma}\,\mathbf{n}=\mathbf{t} \quad \text{on} \quad \Gamma_d, \tag{2}$$

where $\mathbf{n}$ is the unit normal vector to the master side of the discontinuity surface, i.e. normal on $\Gamma_d$ pointing toward $\Omega^+$, and $\mathbf{t}$ is the contact traction acting on the master side of the discontinuity. Note that the contact traction is continuous across $\Gamma_d$. In other words, the contact traction acting on the surface of the slave body is $-\mathbf{t}$, essentially the same with $\mathbf{t}$ but in opposite direction.

The space of kinematically admissible displacement fields (trial functions) $\mathcal{U}$ is defined as

$$\mathcal{U}=\{\mathbf{u} \in \mathscr{H}^1 \,|\, \mathbf{u}=\tilde{\mathbf{u}} \text{ on } \Gamma_u \text{ and } \mathbf{u} \text{ discontinuous on } \Gamma_d\}, \tag{3}$$

and the space of kinematically admissible to zero fields $\delta\mathcal{U}$ (test functions) as

$$\delta\mathcal{U}=\{\delta\mathbf{u} \in \mathscr{H}^1 \,|\, \delta\mathbf{u}=\mathbf{0} \text{ on } \Gamma_u \text{ and } \delta\mathbf{u} \text{ discontinuous on } \Gamma_d\}, \tag{4}$$

where the space $\mathscr{H}^1$ is related to the regularity of the kinematic fields that basically allows for discontinuous functions across the discontinuity [43]. Treating the discontinuity as an external boundary for $\Omega^+$ and $\Omega^-$, the variational formulation associated with the above mentioned contact boundary value problem can be