Finite Elements in Analysis and Design 46 (2010) 838–842

Contents lists available at ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.com/locate/finel

A novel finite element model for vibration analysis of rotating tapered Timoshenko beam of equal strength

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ARTICLE INFO

Article history:
Received 19 May 2009
Accepted 23 May 2010
Available online 8 June 2010

Keywords:
Timoshenko beam
Finite element
Rotating beam
Equal strength

ABSTRACT

A new finite element model based on the coupled displacement field and the tapering functions of the beam is formulated for transverse vibrations of rotating Timoshenko beams of equal strength. In the coupled displacement field, the polynomial coefficients of transverse displacement and cross-sectional rotation are coupled through consideration of the differential equations of equilibrium. The tapering functions of breadth and depth of the beam are obtained from the principle of equal strength in the longitudinal direction of the beam. After finding the displacement functions using the tapering functions, the stiffness and mass matrices are expressed by using the strain and kinetic energy equations. A semi-symbolic computer program in Mathematica is developed and subsequently used to evaluate the new model. The results of the illustrative example regarding the problem indicated in the title of this paper are obtained and compared with the results found from the models created in ABAQUS. Very good agreement is found between the results of new model and the other results.

1. Introduction

The effects of shear deformation and rotary inertia are taken into account in Timoshenko beam theory [1] which is expressed by two coupled partial differential equations. For vibration of tapered beams, the aforementioned two coupled equations have variable coefficients. Furthermore, consideration of the rotational effects on the beam vibrations causes addition of a term with variable coefficient in these equations. Therefore, to find the exact solutions of these equations is generally difficult. Selecting a number of cross-section variation functions, the problem has been solved mainly by numerical or approximate methods such as the dynamic discretization technique [2], the spline interpolation technique [3], the transfer matrix method [4], and the method of Frobenius [5]. A carefully selected sample of the literature on the finite element studies for uniform and tapered rotating/nonrotating Timoshenko beams are given in author's previous study [6] which reports that there is no available shape functions incorporated with the taper parameters based on the breadth and/or depth of the tapered beam. Yardimoglu [6] derived the shape functions, considering the functions of breadth and depth of the tapered beam as power function for vibration analysis of rotating Timoshenko beam.

Hence, the objective of this paper is to present a novel finite element model based on the coupled displacement field incorporating the taper functions of breadth and depth of the Timoshenko beam of equal strength in the longitudinal direction of the beam. The new finite element model has exact stiffness matrix, but approximate mass matrix owing to the usage of static equilibrium condition as in Refs. [7,8]. Petyt [9] reported that the usage of the static equilibrium condition in obtaining an approximate solution for the dynamic response requires an increase in the number of elements needed for a desired accuracy, but this is fully compensated by the simplicity of the mathematical analysis it provides. The new model is verified for out-of-plane vibration of rotating tapered Timoshenko beam of equal strength by comparison of the results obtained from the semi-symbolic computer program developed in Mathematica with the results found from the finite element models created in ABAQUS. In order to show the accuracy of the present new model clearly, the results are given in tabular form.

2. Formulation of the breadth and depth of the beam

Governing equation of equilibrium of a beam under distributed axial load $q(z)$ is given by Bickford [10] as

$$\frac{d}{dz}[A(z)\sigma(z)]+q(z)=0 \tag{1}$$

The notation used throughout this paper is listed in the Nomenclature. For a rotating beam with constant angular velocity Ω, distributed axial load is written as

$$q(z)=\rho A(z)\Omega^2 z \tag{2}$$

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doi:10.1016/j.finel.2010.05.003

Nomenclature

$A(z)$, A_0 cross-sectional area of the beam and its coefficient
$b(z)$, b_0 breadth of the beam and its coefficient
$[B]$ polynomial coefficients coupling matrix
$\{c_v\}$ polynomial coefficient vector of transverse displacement
$\{c_\theta\}$ polynomial coefficient vector of cross-sectional rotation
$[C]$ element displacement-polynomial coefficient matrix
C_1, C_2 constants of integration
$erfi(z)$ imaginary error function of z
E,G elastic modulus and shear modulus, respectively
$h(z)$,h_0 depth of the beam and its coefficient
$I(z)$,I_{xx0} area moments of inertia of the beam cross-section about x and its coefficient
k shear coefficient
$[K]$ global elastic stiffness matrix
$[K_e]$ element elastic stiffness matrix
L length of the beam
m breadth taper parameter
$M(z)$ bending moment about x axis
$[M]$ global mass matrix
$[M_e]$ element mass matrix
n depth taper parameter
N number of element
p taper parameter for area moments of inertia of the cross-section
$P(z)$ centrifugal force
$[P_v]$ polynomial vector for transverse displacement
$[P_\theta]$ polynomial vector for cross-sectional rotation
$q(z)$ distributed axial load
$\{q\}$ global displacement vector
$\{q_e\}$ element displacement vector
r taper parameter for cross-sectional area of the beam
r_g radius of gyration of the root cross-section of the beam about x axis
SF safety factor
$[S]$ global geometric stiffness matrix
$[S_e]$ element geometric stiffness matrix
T kinetic energy
U_e,U_g elastic and geometric strain energies
$v(z,t)$ transverse displacement
$v_0(t)$,$v_1(t)$,$v_2(t)$,$v_3(t)$ polynomial coefficients of the transverse displacement
$V(z)$ shear force in y direction
z_r co-ordinate of the root of the beam
z_t $=z_r+L$ co-ordinate of the tip of the beam
$\theta(z,t)$ cross-sectional rotation about x axis
$\theta_0(t)$,$\theta_1(t)$,$\theta_2(t)$ polynomial coefficients of the cross-sectional rotation
σ_0 $=\sigma_Y/SF$, allowable stress
σ_Y yield stress
$\sigma(z)$ normal stress
ρ density
ω natural circular frequency of beam
Ω rotational speed

In order to obtain the cross-sectional area function of the beam of equal strength, normal stress $\sigma(z)$ in Eq. (1) is considered as constant, namely $\sigma(z)=\sigma_0$. Thus substituting Eq. (2) into Eq. (1), and then solving the equation yields,

$$A(z) = A_0 \exp(-rz^2) \tag{3}$$

where

$$r = 0.5\,\rho\Omega^2/\sigma_0 \tag{4}$$

Now, the breadth and depth of the beam depending on the parameters m and n which satisfy the relation $r=m+n$ are selected as follows:

$$b(z) = b_0 \exp(-mz^2) \tag{5}$$

$$h(z) = h_0 \exp(-nz^2) \tag{6}$$

3. Formulation of the finite element displacement functions

The differential equations of motion for a nonuniform Timoshenko beam are given in Ref. [11]. The homogeneous form of these equations are written as follows:

$$\frac{dM(z)}{dz} + V(z) = 0 \tag{7}$$

$$\frac{dV(z)}{dz} = 0 \tag{8}$$

where

$$M(z) = EI(z)\frac{d\theta(z)}{dz} \tag{9}$$

$$V(z) = kA(z)G\left[\frac{dv}{dz} - \theta(z)\right] \tag{10}$$

Considering the constant shear force given by Eq. (8) in Eq. (7), bending moment is obtained by integration as

$$M(z) = C_1 z + C_2 \tag{11}$$

The cross-sectional rotation of the beam is found by substituting Eq. (11) into Eq. (9), and integrating as

$$\theta(z) = \int \frac{1}{EI(z)}(C_1 z + C_2)dz \tag{12}$$

Transverse displacement of the beam is expressed by substituting Eq. (12) along with Eqs. (9) and (10) into Eq. (7), and integrating as

$$v(z) = \int \left\{\theta(z) - \frac{1}{kA(z)G}\frac{d}{dz}\left[EI(z)\frac{d\theta(z)}{dz}\right]\right\}dz \tag{13}$$

In order to express the displacement functions depending on the functions of the breadth and depth of the beam formulated in Eqs. (5) and (6), the area moment of inertia of the cross-section about x axis can be written as

$$I(z) = I_{xx0} \exp(-pz^2) \tag{14}$$

where $p=m+3\,n$. Then, substituting Eq. (14) into Eq. (12), and integrating yields

$$\theta(z,t) = \theta_0(t) + \theta_1(t) f_1(z) + \theta_2(t) f_2(z) \tag{15}$$

where

$$f_1(z) = \exp(-pz^2) \tag{16a}$$

$$f_2(z) = erfi(\sqrt{p}z) \tag{16b}$$

Also, substituting Eqs. (3), (14), and (15) into Eq. (13), and integrating yields

$$v(z,t) = v_0(t) + v_1(t)g_1(z) + v_2(t)g_2(z) + v_3(t)g_3(z) \tag{17}$$