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A point interpolation method with locally smoothed strain field (PIM-LS2) for mechanics problems using triangular mesh

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ABSTRACT

A point interpolation method with locally smoothed strain field (PIM-LS2) is developed for mechanics problems using a triangular background mesh. In the PIM-LS2, the strain within each sub-cell of a nodal domain is assumed to be the average strain over the adjacent sub-cells of the neighboring element sharing the same field node. We prove theoretically that the energy norm of the smoothed strain field in PIM-LS2 is equivalent to that of the compatible strain field, and then prove that the solution of the PIM-LS2 converges to the exact solution of the original strong form. Furthermore, the softening effects of PIM-LS2 to system and the effects of the number of sub-cells that participated in the smoothing operation on the convergence of PIM-LS2 are investigated. Intensive numerical studies verify the convergence, softening effects and bound properties of the PIM-LS2, and show that the very "tight" lower and upper bound solutions can be obtained using PIM-LS2.

1. Introduction

To solve engineering problems, powerful numerical methods have been developed, such as the Finite Element Methods (FEM) [1–4], Finite Difference Methods [5,6], Finite Volume Methods (FVM) [7,8], and recently Meshfree Methods [9–19]. These methods and techniques not only provide solution tools for many engineering problems but also extend our minds in the quest for even more effective and robust computational methods.

In these methods, the FEM is the most widely used reliable numerical approach for engineering problems. However, the fully compatible FEM based on the standard Galerkin weak form is "overly-stiff", hence resulting in a very poor stress solution, especially when a linear displacement field and a triangular mesh are used [1,2]. Although some higher order and mixed models of FEM are used to obtain good properties, it obviously adds the extra computing cost and computing complexity.

On the other hand, meshfree methods offer attractive alternatives to the FEM for many engineering problems, where the treatments on both field function approximation and integration of the weak form are often different from those in the FEM (see, e.g., [17,18]). In some of the meshfree methods the integration is node-based, and the models are often too "soft", and even spatially unstable. A strain smoothing technique has been applied by Chen et al. [20] to stabilize the nodal integrated Galerkin meshfree methods. By combining the existing FEM and the strain smoothing technique, Liu et al. [21–28] developed some effective methods to provide the softening effects to system for mechanical problems. The node-based smoothed PIM (NS-PIM) is first formulated using PIM shape functions [21]. It was found that NS-PIM can produce much better stress solution, is much more tolerant to mesh distortion, works very well for triangular elements and more importantly it provides upper bound solution in energy norm. Recently, a very effective edge-based smoothed FEM (ES-FEM) [27] has been formulated. The ES-FEM not only produces accurate solution but also is temporally stable and without any spurious modes and hence works very well for dynamic problems.

These works show that the softening effect from smoothing operation will propositionally depend on the number of sub-domains that participated in the smoothing operation. It is known that there are no smoothed operations in the FEM, and the system is "overly-stiff", which results in a lower bound solution in energy norm. In the NS-PIM, the smoothed domain used for strain reconstruction is an entire nodal domain that includes much more sub-cells participated in smooth operation. This results in a very "loose" upper bound solution [28,29] even if a higher order

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displacement model is used. In the ES-PIM, on the other hand, the number of sub-cells that participated in smoothing operation is obviously less than that from NS-PIM. Hence, ES-PIM produces a very high accurate solution. Therefore, the bound property and convergence will be strongly determined by the number of sub-cells that participated in the smoothing operation.

Recently, numerical methods with an adjustable parameter α were developed [29–31] to look for "tight" lower and upper bound solutions. However, the preferable α is usually problem dependent and mesh dependent, and hence requires additional efforts to determine for a practical problem.

To obtain a very tight bound solution and to avoid the difficulty to look for preferable α, we construct a point interpolation method with locally smoothed strain field (PIM-LS2) using a triangular mesh. The aim of this work is to investigate the effects of the number of sub-domains that participated in the smoothing operation on the convergence. Special attention was paid to how to obtain the very "tight" lower and upper bound solutions using PIM-LS2.

The paper is outlined as follows. Section 2 briefs the linear elasticity and NS-PIM. The idea of the PIM-LS2 is presented in Section 3. The convergence and bound properties of the PIM-LS2 are presented and theoretically proven in Section 4. In Section 5, softening effects of PIM-LS2 are discussed, and numerical examples are presented in Section 6. Conclusions are drawn in Section 7.

2. Brief on basic equations

2.1. Brief on basic equations of linearity elasticity [1]

Consider a 2D static elasticity problem governed by the equilibrium equation in the domain Ω bounded by $\Gamma(\Gamma=\Gamma_u+\Gamma_t, \Gamma_u\cap\Gamma_t=0)$ as

$$\mathbf{L}_d^T\boldsymbol{\sigma}+\mathbf{b}=0 \text{ in } \Omega \tag{1}$$

where $\mathbf{L}_d$ is a matrix of differential operators defined as

$$\mathbf{L}_d=\begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_2} \\ 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix}^T \tag{2}$$

where $\boldsymbol{\sigma}^T=(\sigma_{11},\sigma_{22},\sigma_{12})$ is the vector of stresses, $\mathbf{b}^T=(b_1,b_2)$ the vector of body forces. The stresses relate the strains via the generalized Hook's law

$$\boldsymbol{\sigma}=\mathbf{D}\boldsymbol{\varepsilon} \tag{3}$$

where $\mathbf{D}$ is the matrix of material constants [17] and $\boldsymbol{\varepsilon}^T=(\varepsilon_{11},\varepsilon_{22},2\varepsilon_{12})$ is the vector of strains given by

$$\boldsymbol{\varepsilon}=\mathbf{L}_d\mathbf{u} \tag{4}$$

Essential boundary conditions are

$$\mathbf{u}=\mathbf{u}_0 \text{ on } \Gamma_u \tag{5}$$

where $\mathbf{u}^\mathbf{T}=(u_1,u_2)$ is the vector of the displacement and $\mathbf{u}_0$ is the vector of the prescribed displacements on the essential boundary Γ_u. In this paper, we consider only homogenous essential boundary conditions $\mathbf{u}_0=0$.

Natural boundary conditions are

$$\mathbf{L}_n^T\boldsymbol{\sigma}=\mathbf{T} \text{ on } \Gamma_t \tag{6}$$

where $\mathbf{T}$ is the vector of the prescribed tractions on the natural boundary Γ_t, and $\mathbf{L}_n$ is the matrix of unit outward normal, which can be expressed as

$$\mathbf{L}_n=\begin{bmatrix} n_{x_1} & 0 & n_{x_2} \\ 0 & n_{x_2} & n_{x_1} \end{bmatrix}^T \tag{7}$$

2.2. Briefing on the NS-PIM [28]

In the node-based smoothed PIM (NS-PIM), the problem domain is first discretized by a set of background triangular cells as shown in Fig. 1. The displacements in a cell are approximated using PIM shape functions

$$\bar{\mathbf{u}}(\mathbf{x})=\sum_{i\in n_e}\boldsymbol{\Phi}_i(\mathbf{x})\bar{\mathbf{d}}_i \tag{8}$$

where n_e is the set of nodes of the local support domain containing $\mathbf{x}$ which is in general beyond the cell, $\bar{\mathbf{d}}_i$ is a vector of displacements at this set of nodes, and

$$\boldsymbol{\Phi}_i(\mathbf{x})=\begin{bmatrix} \varphi_i(\mathbf{x}) & 0 \\ 0 & \varphi_i(\mathbf{x}) \end{bmatrix} \tag{9}$$

is the matrix of the shape function for node i, which is constructed generally using the PIM procedure and hence is of Delta function property.

By connecting sequentially the mid-edge-point of a background cell to its centroids, the problem domain Ω is divided into smoothing domains Ω_k containing node k, as shown in Fig. 1. NS-PIM uses constant strain for each smoothing domain as follows [20]:

$$\bar{\boldsymbol{\varepsilon}}_k\equiv\bar{\boldsymbol{\varepsilon}}(\mathbf{x}_k)=\frac{1}{A_k}\int_{\Omega_k}\tilde{\boldsymbol{\varepsilon}}(\mathbf{x})d\Omega \tag{10}$$

where $A_k=\int_{\Omega k}d\Omega$ is the area of smoothing domain for node k, and $\tilde{\boldsymbol{\varepsilon}}(\mathbf{x})=\mathbf{L}_d\bar{\mathbf{u}}$ is the compatible strain.

The assumed displacement $\bar{\mathbf{u}}$ and the corresponding assumed strains $\bar{\boldsymbol{\varepsilon}}$ satisfy the generalized smoothed Galerkin weak form

$$\int_\Omega\delta\bar{\boldsymbol{\varepsilon}}^T(\bar{\mathbf{u}})\mathbf{D}\bar{\boldsymbol{\varepsilon}}(\bar{\mathbf{u}})d\Omega-\int_\Omega\delta\bar{\mathbf{u}}^T\mathbf{b}\,d\Omega-\int_{\Gamma_t}\delta\bar{\mathbf{u}}^T\mathbf{T}\,d\Gamma=0 \tag{11}$$

Substituting Eqs. (8) and (10) into Eq. (11) yields the discretized system equation

$$\bar{\mathbf{K}}\ \bar{\mathbf{d}}=\bar{\mathbf{f}} \tag{12}$$

Remark 1. (*Upper bound property of NS-PIM* [28,29]): For any practical model with a reasonable number of elements, the strain energy obtained using the NS-PIM is not less than the exact strain energy.

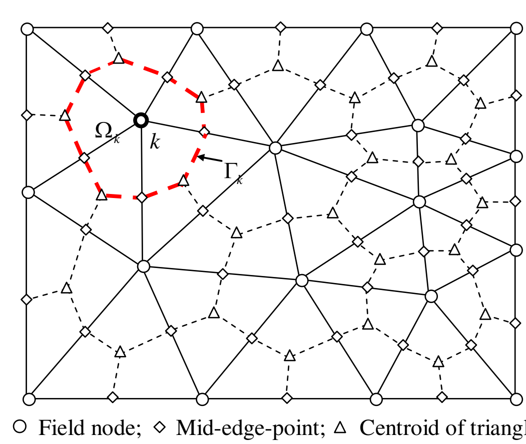

Fig. 1. Triangular background elements and the nodal smoothing domains in NS-PIM.