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Modified fixed-pole approach in geometrically exact spatial beam finite elements

M. Gaćeša, G. Jelenić *

University of Rijeka, Faculty of Civil Engineering, Radmile Matejčić 3, 51000 Rijeka, Croatia

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ABSTRACT

A family of spatial beam finite elements based on the fixed-pole approach is developed and presented in this work. The family consists of three different interpolation options, which arise as a consequence of the fact that the virtual position vector has been interpolated in a non-linear manner, and depends on the actual position vector not only at the nodal points, but also at the integration points. All three formulations turn out to be non-invariant and path-dependent. In two of them, however, this problem may be easily solved by interpolating the total local rotations, which also makes the procedure more robust in handling large load increments.

1. Introduction and motivation

The topic of this paper is the development of a family of spatial beam finite elements based on the geometrically exact beam theory given by Reissner [1] for 2D beams and extended to 3D beams by Simo [2]. Implementation of this theory in a finite-element framework becomes non-trivial because of the properties of rotations in 3D. These are defined by orientation matrices, elements of a special orthogonal group SO(3), which is also a Lie group (see e.g. [3]). By definition, a Lie group is also a differentiable manifold, and in order for the solution to stay on the manifold, we must acknowledge existence of the so-called exponential map, a specific operation between a Lie group and a related Lie algebra (in this case the algebra of 3D skew-symmetric tensors so(3)). This theory was first implemented by Simo and Vu-Quoc, interpolating the infinitesimal rotations, the so-called spin vectors [4]. Similar results can be obtained interpolating the incremental rotational changes (between two converged configurations) as proposed by Cardona and Géradin [5], or the total rotations (the change between the initial and current configuration) as given in [6]. The additive interpolation of any of these rotational fields was shown to be the source of non-invariance of strain measures with respect to a rigid motion [7], while Romero [8] thoroughly examined and discussed how different interpolations affect the invariance properties.

An invariant formulation was proposed in [9] which not only solved the problem of non-invariance of the strain measures but also the problem of path dependence which also existed in some of those formulations, by interpolating only the relative rotations between specific nodes i.e. the rotations from which the rigid-rotation is removed. In a different vein, Betsch and Steinmann [10] have proposed a formulation where interpolation of the rotational degrees of freedom is circumvented by reparametrising the orientation matrix using and interpolating its base triad, while Zupan and Saje [11] developed a finite element where the strain vectors are interpolated.

In this work we want to shed more light on the fascinating and highly promising fixed-pole concept, and investigate alternative options of its implementation into the geometrically exact 3D beam theory to those currently available. In the context of this theory, the fixed-pole concept was first introduced by Borri and Bottasso in 1994 [12] and thoroughly researched in a series of subsequent papers [13–16]. In the context of simple-material and polar-material elastomechanics, the concept was presented in [17,18].

It appears that Borri and Bottasso were concerned with modelling curved mechanical elements such as wind or helicopter blades, for which it ceased to be obvious how to define a "proper" reference axis. In [12,13] they approached this problem by interpolating the kinematic quantities along the beam arc-length: the authors assumed that the reference axis of the beam element had a shape of a spatial helix and that both the translational and the rotational strain measures along it should be constant. This resulted in a so-called *helicoidal* interpolation. Although it does not appear that this was their chief motivation such interpolation also solved the problem of non-invariance of the strain measures but it was naturally applied only to two-node elements [19]. Their idea has been recently explored and generalised to an element of arbitrary order by Papa Dukić et. al. [20].

* Corresponding author.
E-mail address: gordan.jelenic@uniri.hr (G. Jelenić).

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In [14–16] Borri and Bottasso with co-workers managed to merge the displacement and the rotation fields and introduced a *configuration tensor* which uniquely determined both the position and the orientation of a cross-section under consideration. This tensor turned out to be an element of the special group of rigid motions SR(6), a matrix representation of the special Euclidean group SE(3) [15,16] and a Lie group. They further proved the existence of a closed form of the exponential map between SR(6) and the related Lie algebra sr(6), which, in contrast to the earlier helicoidal interpolation, they now interpolated using Lagrangian polynomials. By employing all this, they obtained an extremely elegant formulation which used a unique operation for updating the kinematic fields, which in turn appeared to be particularly suitable for simultaneous conservation of energy and momenta in conservative time-stepping integrators. The complexities associated with simultaneous conservation of energy and momenta in 3D beams were explored in [21–24]. Recently, Sonneville and co-workers [25] proposed a formulation for static and dynamic analysis of geometrically exact beams which combined the ideas of the configuration tensor and helicoidal interpolation. Their particular algorithm was based on a 4D matrix representation of SE(3) and a helicoidal interpolation [12].

We emphasise the two distinct novelties proposed by Borri, Bottasso and their co-workers: the *helicoidal interpolation* and the *configuration-tensor approach*. Although they may go together, as they do in [25], there is no reason not to address (and implement) them separately. In our recent work [20] we have proposed a possible generalisation of the helicoidal interpolation [12] to higher-order elements without any reference to the configuration-tensor aspect of the fixed-pole approach. In the present work, in contrast, we focus on this aspect only and investigate possible improvements. In particular, the system unknowns used in the fixed-pole approach using the configuration-tensor representation are *non-standard*, and as a consequence, (i) the finite elements based on this approach cannot be combined with meshes of standard elements (with displacements and rotations as the system unknowns) and, (ii) the boundary conditions also have to be imposed in a specific non-standard manner.

In the present paper, we present a family of novel 3D beam finite elements based on the fixed-pole concept with the standard degrees of freedom (displacements and rotations) as the nodal unknowns. We investigate these elements with respect to their strain-invariance and path-independence properties and propose a method of designing reliable and well-behaving elements.

2. Background results

2.1. Simo and Vu-Quoc approach [4]

The Reissner–Simo [1,2] non-linear beam theory may be derived from the principle of virtual work which states that the virtual work done by internal and inertial forces must be equal to the work done by applied external loading. The translational and rotational material strain measures are defined as

$$\mathbf{\Gamma} = \mathbf{\Lambda}^T\mathbf{r}' - \mathbf{E}_1 \tag{1}$$

$$\widehat{\mathbf{K}} = \mathbf{\Lambda}^T\mathbf{\Lambda}', \tag{2}$$

with $\mathbf{r}$ as the position vector of a cross section, $\mathbf{E}_1$ as the unit vector along the beam reference axis in the material coordinate system and $\mathbf{\Lambda} \in SO(3)$ as the rotation matrix of a cross-section with respect to the origin of the spatial orthogonal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and, for $\mathbf{E}_i = \mathbf{e}_i$, also a linear mapping between the spatial and the material objects. It enjoys the properties of unimodularity and orthogonality while, for any 3D vector $\mathbf{a}$, the hat operator $\widehat{\mathbf{a}}$ defines a skew-symmetric matrix operating on another 3D vector $\mathbf{b}$ as a cross-product $\widehat{\mathbf{a}}\mathbf{b} = \mathbf{a}\times\mathbf{b}$. As a result, $\widehat{\mathbf{a}}^T = -\widehat{\mathbf{a}}$, $\widehat{\mathbf{a}}\mathbf{b} = -\widehat{\mathbf{b}}\mathbf{a}$, $\widehat{\mathbf{a}\times\mathbf{b}} = \widehat{\mathbf{a}}\widehat{\mathbf{b}} - \widehat{\mathbf{b}}\widehat{\mathbf{a}}$ and $\widehat{\mathbf{\Lambda}\mathbf{a}} = \mathbf{\Lambda}\widehat{\mathbf{a}}\mathbf{\Lambda}^T$. The material stress resultants $\mathbf{N}$, $\mathbf{M}$ are, for a linear elastic material, related to the material strain measures via $\mathbf{N} = \mathbf{C}_N\mathbf{\Gamma}$ and $\mathbf{M} = \mathbf{C}_M\mathbf{K}$ with $\mathbf{C}_N = \mathrm{diag}[EA_1\ GA_2\ GA_3]$ and $\mathbf{C}_M = \mathrm{diag}[GJ\ EI_2\ EI_3]$ as the translational and rotational constitutive matrices. Here, A_1, A_2, A_3 denote the cross-sectional and shear areas and J, I_2, I_3 denote the torsional constant and cross-sectional second moments of area, while G and E are the shear and Young's modulus, respectively.

The material triad $\mathbf{E}_i$ and the moving triad $\mathbf{t}_i$ rigidly attached to the beam reference axis are related via $\mathbf{t}_i = \mathbf{\Lambda}\mathbf{E}_i$. We can obtain the spatial strain measures by mapping the material strain measures into the ambient space via $\mathbf{\Lambda}$. In this way

$$\boldsymbol{\gamma} = \mathbf{\Lambda}\mathbf{\Gamma} = \mathbf{r}' - \mathbf{t}_1, \tag{3}$$

$$\widehat{\boldsymbol{\kappa}} = \mathbf{\Lambda}\widehat{\mathbf{K}}\mathbf{\Lambda}^T = \mathbf{\Lambda}'\mathbf{\Lambda}^T \Rightarrow \boldsymbol{\kappa} = \mathbf{\Lambda}\mathbf{K} \tag{4}$$

and $\mathbf{n} = \mathbf{\Lambda}\mathbf{N}$ and $\mathbf{m} = \mathbf{\Lambda}\mathbf{M}$. The virtual work of internal forces is

$$V_i = \int_0^L (\delta\mathbf{\Gamma}\cdot\mathbf{N} + \delta\mathbf{K}\cdot\mathbf{M})\,\mathrm{d}x = \int_0^L \left(\overset{\triangledown}{\delta\boldsymbol{\gamma}}\cdot\mathbf{n} + \overset{\triangledown}{\delta\boldsymbol{\kappa}}\cdot\mathbf{m}\right). \tag{5}$$

The virtual change of the translational strain measure (1) follows as

$$\delta\mathbf{\Gamma} = \delta\left(\mathbf{\Lambda}^T\mathbf{r}' - \mathbf{E}_1\right) = \mathbf{\Lambda}^T(\delta\mathbf{r}' + \mathbf{r}'\times\delta\boldsymbol{\vartheta}),$$

since the virtual change of $\mathbf{\Lambda}$ follows as $\delta\mathbf{\Lambda} = \widehat{\delta\boldsymbol{\vartheta}}\mathbf{\Lambda}$, with $\delta\boldsymbol{\vartheta}$ as the spatial spin vector (see [2] for details). The virtual change of the rotational strain measure (2) follows as

$$\delta\widehat{\mathbf{K}} = \delta\left(\mathbf{\Lambda}^T\mathbf{\Lambda}'\right) = \mathbf{\Lambda}^T\delta\widehat{\boldsymbol{\vartheta}}'\mathbf{\Lambda} \Rightarrow \delta\mathbf{K} = \mathbf{\Lambda}^T\delta\boldsymbol{\vartheta}',$$

while the objective spatial strain rates are

$$\begin{aligned} \overset{\triangledown}{\delta\boldsymbol{\gamma}} &= \delta\boldsymbol{\gamma} + \boldsymbol{\gamma}\times\delta\boldsymbol{\vartheta} = \delta\mathbf{r}' + \mathbf{r}'\times\delta\boldsymbol{\vartheta} = \mathbf{\Lambda}\delta\mathbf{\Gamma}, \\ \overset{\triangledown}{\delta\boldsymbol{\kappa}} &= \delta\boldsymbol{\kappa} + \boldsymbol{\kappa}\times\delta\boldsymbol{\vartheta} = \delta\boldsymbol{\vartheta}' = \mathbf{\Lambda}\delta\mathbf{K}, \end{aligned} \tag{6}$$

where obviously $\overset{\triangledown}{\delta\boldsymbol{\gamma}}\cdot\mathbf{n} + \overset{\triangledown}{\delta\boldsymbol{\kappa}}\cdot\mathbf{m} \neq \delta\boldsymbol{\gamma}\cdot\mathbf{n} + \delta\boldsymbol{\kappa}\cdot\mathbf{m}$. In other words, while $(\mathbf{\Gamma}, \mathbf{K})$ and $(\boldsymbol{\gamma}, \boldsymbol{\kappa})$ are both the respective strain-energy conjugates to $(\mathbf{N}, \mathbf{M})$ and $(\mathbf{n}, \mathbf{m})$, i.e. $\phi = \frac{1}{2}\int_0^L (\mathbf{\Gamma}\cdot\mathbf{N} + \mathbf{K}\cdot\mathbf{M})\,\mathrm{d}x = \frac{1}{2}\int_0^L (\boldsymbol{\gamma}\cdot\mathbf{n} + \boldsymbol{\kappa}\cdot\mathbf{m})\,\mathrm{d}x$, only $(\delta\mathbf{\Gamma}, \delta\mathbf{K})$ are the virtual-work conjugates to $(\mathbf{N}, \mathbf{M})$, while the virtual-work conjugates to $(\mathbf{n}, \mathbf{m})$ are $(\overset{\triangledown}{\delta\boldsymbol{\gamma}}, \overset{\triangledown}{\delta\boldsymbol{\kappa}})$ rather than $(\delta\boldsymbol{\gamma}, \delta\boldsymbol{\kappa})$.

The virtual work of inertial forces is $V_m = \int_0^L \left(\delta\mathbf{r}\cdot\dot{\mathbf{k}} + \delta\boldsymbol{\vartheta}\cdot\dot{\boldsymbol{\pi}}\right)\mathrm{d}x$ with $\mathbf{k} = A\rho\dot{\mathbf{r}}$ and $\boldsymbol{\pi} = \mathbf{\Lambda}\mathbf{J}_\rho\mathbf{W}$ as the translational and angular momenta per unit of length, where a superimposed dot denotes a derivative with respect to time, $A\rho$ is the beam mass per unit of length, $\mathbf{J}_\rho = \mathrm{diag}[J_1, J_2, J_3]$ is the material tensor of moments of inertia per unit of length and $\mathbf{W}$ is the material angular velocity, where $\widehat{\mathbf{W}} = \mathbf{\Lambda}^T\dot{\mathbf{\Lambda}}$.

Finally, the virtual work of external loading is $V_e = \int_0^L (\delta\mathbf{r}\cdot\mathbf{n}_e + \delta\boldsymbol{\vartheta}\cdot\mathbf{m}_e)\,\mathrm{d}x + \delta\mathbf{r}_1\cdot\mathbf{F}_0 + \delta\boldsymbol{\vartheta}_1\cdot\mathbf{M}_0 + \delta\mathbf{r}_N\cdot\mathbf{F}_L + \delta\boldsymbol{\vartheta}_N\cdot\mathbf{M}_L$, with $\mathbf{n}_e, \mathbf{m}_e$ as the forces and moments per unit of length and $\mathbf{F}_0, \mathbf{M}_0, \mathbf{F}_L, \mathbf{M}_L$ as the end-point forces and moments.

Substituting V_i, V_m and V_e in the principle of virtual work and interpolating $\delta\mathbf{r}$ and $\delta\boldsymbol{\vartheta}$ using Lagrangian interpolation results in the nodal residual defined as

$$\mathbf{g}^i = \mathbf{q}_i^i + \mathbf{q}_m^i - \mathbf{q}_e^i = \mathbf{0}. \tag{7}$$

Here,

$$\mathbf{q}_i^i = \int_0^L \begin{bmatrix} I^{i'}\mathbf{I} & \mathbf{0} \\ -I^i\widehat{\mathbf{r}'} & I^{i'}\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{n} \\ \mathbf{m} \end{Bmatrix}\mathrm{d}x, \tag{8}$$