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# Efficient force method for the analysis of finite element models comprising of triangular elements using ant colony optimization

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ABSTRACT

An efficient algorithm is presented for the formation of null basis of triangular plane stress and plane strain finite element models, corresponding to highly sparse flexibility matrices. This is achieved by applying a modified ant colony system (ACS). An integer linear programming formulation is also presented to evaluate the quality of the results obtained by the proposed ant colony system algorithm. The efficiency of the present algorithm is illustrated through some examples.



## 1. Introduction

The force method of structural analysis, in which the member forces are used as unknowns, is appealing to engineers, since the properties of members of a structure most often depend on the member forces rather than joint displacements.

Four different approaches are adopted for the force method of structural analysis, which can be classified as: (1) topological force methods, (2) algebraic force methods, (3) mixed algebraic–topological force methods, and (4) integrated force method.

Topological methods have been developed by Henderson and Maunder [1] and Maunder [2] for rigid-jointed skeletal structures using manual selection of cycle bases. Methods suitable for computer programming are due to Kaveh [3,4]. Algebraic methods have been developed by Denke [5], Robinson [6], Topçu [7], Kaneko et al. [8], and mixed algebraic–topological methods have been used by Gilbert and Heath [9], Coleman and Pothen [10,11]. The integrated force method has been developed by Patnaik [12] and Patnaik et al. [13], in which the equilibrium equations and compatibility conditions are satisfied simultaneously in terms of the force variables.

The force method of structural analysis requires the formation of a maximal set of independent self-equilibrating stress systems (SESs), known as a *null basis* [14,15]. The elements of this basis form the columns of an $m\times\gamma(S)$ matrix, $B_1$, known as the *self-stress matrix*.

The main problem in the application of the force method is the formation of a self-stress matrix corresponding to a sparse flexibility matrix $G=B_1^t F_m B_1$, where $F_m$ contains the flexibility matrices of the individual members of the structure in a block diagonal form.

The combinatorial methods for the force method are very efficient for skeletal structures and, in particular, for rigid-jointed frames. For a general structure, the underlying graph or hypergraph of a SES has not yet been properly defined, and further research is needed. Algebraic methods, on the other hand, can be formulated in a more general form to cover different types of structures such as skeletal structures and finite element models (FEM). The main drawbacks of these methods are the large storage requirements and the higher number of operations.

Heuristic algorithms, such as ant colony algorithms, have found many applications in optimization problems in the last decade. The essence of these algorithms lies in the fact that their capability to converge to a good solution does not depend on the specific search space to which they are applied. In this paper, the ant colony system (ACS) which is a variation of the ant colony optimization (ACO) is applied to the formation of null bases of triangular plane stress and plane strain finite element models corresponding to highly sparse and banded flexibility matrices. An integer linear programming formulation is presented to evaluate the quality of the results obtained

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by the proposed ACS algorithm. The efficiency of the present method is illustrated through simple examples.

## 2. Formulation of the force method

Consider a structure $S$, which is $\gamma(S)$ times statically indeterminate. $\gamma(S)$ independent unknown forces are selected as redundants. These unknown forces can be selected from external reactions and or internal forces of the structure. These redundants are denoted by a vector as $q=\{q_1,q_2,\dots,q_{\gamma(S)}\}^t$. In order to obtain a statically determinate structure, known as the *basic* (released or primary) *structure* of $S$, the constraints corresponding to redundants are removed. Consider the joint loads vector as $p=\{p_1,p_2,\dots,p_n\}^t$, where $n$ is number of entries of the applied nodal load vector and let $r$ denote the $m$-dimensional vector of generalized independent element forces. The equilibrium conditions of the structure can then be expressed as

$$Ar=p \tag{1}$$

where $A$ is an $n\times m$ equilibrium matrix. The element forces can be written as

$$r=B_0p+B_1q \tag{2}$$

where $B_0$ and $B_1$ are rectangular matrices each having $m$ rows, and $n$ and $\gamma(S)$ columns, respectively. $B_1$ is called a *self-stress matrix* as well as *null basis matrix*. Each column of $B_1$ is known as a *null vector*.

Minimizing the complementary potential energy requires that $r$ minimize the quadratic form, $\frac{1}{2}r^tF_mr$ subjected to the constraint of the equilibrium conditions as in Eq. (1). $F_m$ is a $m\times m$ block diagonal element flexibility matrix. Using Eq. (2), it can be seen that $q$ must satisfy the following equation:

$$(B_1^tF_mB_1)q=-B_1^tF_mB_0p \tag{3}$$

where $B_1{}^tF_mB_1=G$ is the *overall flexibility matrix* of the structure. Computing the redundant forces $q$ from Eq. (3), $r$ can be found using Eq. (2), i.e.

$$r=[B_0-B_1(B_1^TF_mB_1)^{-1}B_1^TF_mB_0]p \tag{4}$$

The structure of the matrix $G$ is again important, and its sparsity, bandwidth and conditioning govern the efficiency of the force method. For the sparsity of $G$ one can search for a sparse $B_1$ matrix, which is often referred to as the sparse null basis problem.

## 3. Constant stress triangular element

For this element, the element forces, $F_i=\{F_{\alpha i},F_{\beta i},F_{\gamma i}\}$, are taken as the natural forces acting along the side of the triangles, as shown in Fig. 1. The corresponding displacements are denoted by $\upsilon_i=\{\upsilon_{\alpha i},\upsilon_{\beta i},\upsilon_{\gamma i}\}$.

In a global coordinate system, the nodal forces for each element have six components and the nodal forces and element forces can be related by projection.

The simple flexibility matrix is as

$$f_i=\frac{1}{tA}\mathbf{l}\phi_{ct}\mathbf{l} \tag{5}$$

where $\mathbf{l}=\{L_\alpha,L_\beta,L_\gamma\}$, $A$ is the area of the element, $t$ is the thickness, and

$$\phi_{ct}=\frac{1}{2G}\left[\begin{bmatrix}1&\cos^2\gamma&\cos^2\beta\\ \cos^2\gamma&1&\cos^2\alpha\\ \cos^2\beta&\cos^2\alpha&1\end{bmatrix}+\frac{\upsilon}{1+\upsilon}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\right] \tag{6}$$

## 4. Mathematical modeling for optimization problem

Since the *overall flexibility matrix* of the structure $G$ is $B_1^tF_mB_1$, for the sparsity of $G$ one should select a null basis corresponding to sparse $B_1$ matrix, which is often referred to as the sparse null basis problem. The main objective of this paper is to find sparse self-stress matrices to simplify the solution and to ensure the formation of well-conditioned flexibility matrices.

For a SES (null vector), no applied load is required, thus the equilibrium conditions can be expressed as

$$AB_1=0 \tag{7}$$

This equation shows that the columns of the matrix $A$, which is an $n\times m$ matrix with rank of $n$ are linearly dependent. There are $m-n=t$ independent columns of $B_1$ which will satisfy this equation, thus forming a set of SESs as a basis.

A fact to emphasize is that there are many sets of SESs, which have independent columns and satisfy the above equation. However, the problem is to find a set corresponding to highly sparse $B_1$ matrix.

Let us denote the columns of matrix $B_1$ by $S_i$ as

$$B_1=[S_1,S_2,\dots,S_g,\dots,S_t] \tag{8}$$

Suppose the first null vector $S_1$ is found, then it can be normalized by the following equation:

$$e_1^tS_1=1 \tag{9}$$

where $e_1=\{1\ 0\ \dots\ 0\ \dots\ 0\}$ is an $m\times 1$ unit vector with 1 in the first entry position. The second column $S_2$ can be normalized and must be independent of $S_1$ and these conditions are expressed as

$$e_1^tS_2=0 \tag{10}$$

$$e_2^tS_2=1 \tag{11}$$

where $e_2=\{0\ 1\ 0\ \dots\ 0\ \dots\ 0\}$ is an $m\times 1$ unit vector with 1 in the second entry position. It is obvious that the conditions analogous to these relationships can be formed for the subsequent null vectors.

In this section, first the mathematical programming is employed for selecting the column $S_2$ and then extended for the formation of the complete set of the SESs. The first null vector $S_1$, is arbitrary. Now we find the second null vector $S_2$, satisfying the following equations:

$$AS_2=0 \tag{12}$$

$$e_1^tS_2=0 \tag{13}$$

$$e_2^tS_2=1 \tag{14}$$

or more concisely

$$\begin{bmatrix}A\\ I_2\ \ 0\end{bmatrix}S_2=\begin{bmatrix}0\\ \bar{e}_2\end{bmatrix} \tag{15}$$

where $\bar{e}_2=\{01\}$ is a $2\times 1$ unit vector, with 1 in the $g$th position which minimizes the function $Z=|S_2|$. Here, $|S_2|$ denotes the cardinality of $S_2$ and it is equal to the number of non-zero entries of $S_2$.

This can be generalized for the $g$th null vector $S_g$, after all the previous null vectors up to $g-1$ have been obtained. The problem can now be stated as follows:

Minimize the objective function of the form $Z=|S_g|$ satisfying

$$\begin{bmatrix}A\\ I_g\ \ 0\end{bmatrix}S_g=\begin{bmatrix}0\\ \bar{e}_g\end{bmatrix} \tag{16}$$

where $\bar{e}_g=\{0\ 0\ \dots\ 0\ \dots 1\}$ is a $g\times 1$ unit vector, with 1 in the $g$th entry position.