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Least-squares stabilized augmented Lagrangian multiplier method for elastic contact

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ABSTRACT

In this paper, we propose a stabilized augmented Lagrange multiplier method for the finite element solution of small deformation elastic contact problems. We limit ourselves to friction-free contact with a rigid obstacle, but the formulation is readily extendable to more complex situations.

1. Introduction

This paper is motivated by the recent work by Chouly and Hild [9] on Nitsche's method for contact problems. In their work, the contact conditions were incorporated into the bilinear form to transform the variational inequality describing the contact problem to a nonlinear variational equality. We here point out that the same can be done for a more standard stabilized Lagrange multiplier method if we augment the Lagrangian using the standard approach of, e.g., Alart and Curnier [1]. Using a multiplier method has an advantage compared to the Nitsche method in that there is an increased freedom in choosing the multiplier space. For example, using a continuous multiplier with nodes coinciding with the displacement nodes on the surface we can use nodal quadrature schemes to emulate point Lagrange multipliers (at least for low order elements). For such schemes, contact will be checked at the nodes as is usually done in engineering practice. This is not possible following the Nitsche approach.

A basic issue when using Lagrange multipliers to enforce contact is the number of degrees of freedom in the discrete Lagrange multiplier space. If too many constraints are used, the discrete system might be singular, and if there are too few constraints, there might be unphysical violation of the non-penetration condition. There are, basically, two different possibilities to obtain a stable discretization. The first approach is to choose discrete spaces that fulfill the *inf–sup* condition which guarantees stability (cf. [7]). A well-known example of such a scheme is the mortar method introduced by Bernardi et al. [5], and applied to contact problems by Ben Belgacem et al. [4]. The other option is to change the bilinear form in such a way that stability is ensured, as pioneered by Barbosa and Hughes [2,3] and this is the approach taken in this paper. We consider an augmented version of the stabilized Lagrange multiplier method introduced by Hansbo et al. [11] and adapted to contact by Heintz and Hansbo [12]. Unlike the method in [11,12], which was based on a global polynomial approximation of the multiplier, the method herein is suitable for locally defined multipliers.

The rest of the paper is organized as follows. First, we describe the proposed method. Secondly, we make some comments on the stability and convergence properties of the discretization. Finally, we present some numerical results and conclusions.

2. Problem formulation

We consider an elastic body covering the domain Ω in $\mathbb{R}^d$, $d=2, 3$, with boundary $\Gamma = \Gamma^{\mathrm{D}} \cup \Gamma^{\mathrm{N}} \cup \Gamma^{\mathrm{C}}$ and outward pointing normal $\boldsymbol{n}$. We consider the case where the domains are subjected to proper Dirichlet (on Γ^{D}) and traction (on Γ^{N}) boundary conditions and are coming into frictionless contact along Γ^{C}, and are subjected to volume forces $\boldsymbol{f} \in (L_2(\Omega))^d$. The unilateral contact problem in linear elasticity consists in finding the displacement field $\boldsymbol{u}$

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satisfying the equations and conditions

$$\left.\begin{array}{ll} -\nabla\cdot\boldsymbol{\sigma}=\boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{\sigma}=2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})+\lambda\nabla\cdot\boldsymbol{u}\boldsymbol{I} & \text{in } \Omega, \\ \boldsymbol{u}=\boldsymbol{0} & \text{on } \Gamma^{\mathrm{D}}, \\ \boldsymbol{\sigma}\cdot\boldsymbol{n}=\boldsymbol{0} & \text{on } \Gamma^{\mathrm{N}}, \\ u_n\le 0,\ \sigma_n(\boldsymbol{u})\le 0,\ u_n\,\sigma_n(\boldsymbol{u})=0 & \text{on } \Gamma^{\mathrm{C}}, \end{array}\right\} \tag{1}$$

with $\sigma_n(\boldsymbol{u}) := \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n}$, $u_n := \boldsymbol{u}\cdot\boldsymbol{n}$, and with $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the strain tensor with components

$$\boldsymbol{\varepsilon}(\boldsymbol{u})=\tfrac{1}{2}\left(\nabla\otimes\boldsymbol{u}+(\nabla\otimes\boldsymbol{u})^{\mathrm{T}}\right),$$

where $(\boldsymbol{w}\otimes\boldsymbol{v})_{ij}=w_i v_j$, with $\mu=\frac{E}{2(1+\nu)}$ and $\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)}$, where E is the modulus of elasticity, ν is Poisson's ratio, and with $\boldsymbol{I}$ the identity tensor. We shall assume that μ and λ are constant in the domain.

2.1. An augmented Lagrangian method

One interesting way of deriving an augmented Lagrangian method is to replace $\sigma_n(\boldsymbol{u})$ by a Lagrange multiplier p (the contact pressure) so that the final line in (1) is written

$$u_n\le 0,\quad p\le 0,\quad u_n p=0, \tag{2}$$

(the Kuhn–Tucker conditions) and, using the notation,

$$[a]_+ := \begin{cases} a & \text{if } a>0, \\ 0 & \text{if } a\le 0, \end{cases} \tag{3}$$

replace conditions (2) by the equivalent statement

$$p=-\frac{1}{\gamma}[u_n-\gamma p]_+ \tag{4}$$

with γ a positive number, cf. Chouly and Hild [9, Proposition 2.1].

Defining function spaces

$$V=\{\boldsymbol{v}\in(H^1)^d : \boldsymbol{v}=\boldsymbol{0} \text{ on } \Gamma^{\mathrm{D}}\},\quad Q=L_2(\Gamma^{\mathrm{C}}), \tag{5}$$

and seeking $(\boldsymbol{u},p)\in V\times Q$ we have by Green's theorem, with

$$a(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\Omega,\quad L(\boldsymbol{v}) := \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\Omega,$$

that

$$a(\boldsymbol{u},\boldsymbol{v})-\int_{\Gamma^{\mathrm{C}}} p\,v_n\,\mathrm{d}s=L(\boldsymbol{v})$$

where $\boldsymbol{v}\in V$ and $v_n := \boldsymbol{v}\cdot\boldsymbol{n}$. Following [9] we write $v_n=v_n+\gamma q-\gamma q$ for an arbitrary function $q\in Q$, so that we may write

$$a(\boldsymbol{u},\boldsymbol{v})-\int_{\Gamma^{\mathrm{C}}} p\,(v_n-\gamma q)\,\mathrm{d}s-\int_{\Gamma^{\mathrm{C}}}\gamma p\,q\,\mathrm{d}s=L(\boldsymbol{v}).$$

Replacing the p in the first integral by the expression in (4) we finally obtain the problem of finding $(\boldsymbol{u},p)\in V\times Q$ such that

$$a(\boldsymbol{u},\boldsymbol{v})+\int_{\Gamma^{\mathrm{C}}}\frac{1}{\gamma}[u_n-\gamma p]_+(v_n-\gamma q)\,\mathrm{d}s-\int_{\Gamma^{\mathrm{C}}}\gamma p\,q\,\mathrm{d}s=L(\boldsymbol{v})\quad\forall(\boldsymbol{v},q)\in V\times Q. \tag{6}$$

This problem is related to seeking stationary points to the functional

$$\Pi(\boldsymbol{u},p) := a(\boldsymbol{u},\boldsymbol{u})-L(\boldsymbol{u})+\int_{\Gamma^{\mathrm{C}}}\frac{1}{2\gamma}[u_n-\gamma p]_+^2\,\mathrm{d}s-\int_{\Gamma^{\mathrm{C}}}\frac{\gamma}{2}p^2\,\mathrm{d}s, \tag{7}$$

which is the well known nonlinear variational equality version of the *augmented Lagrangian method*, see, e.g., Alart and Curnier [1]. While replacing the standard variational inequality formulation (cf., e.g., [4,13]) by a nonlinear variational equality is not strictly necessary in an augmented method, cf. Chen [8], the nonlinear equality approach is well suited for applying Newton methods, and allows for avoiding the activation/deactivation of multipliers as the contact zone changes during nonlinear iterations. In a standard variational inequality setting, an active set of multipliers is typically used, which favours iterative algorithms able to accomodate active sets, cf. [6].

The formulation (7) constitutes the starting point for our finite element approximation. A remarkable fact is that by replacing p by $\sigma_n(\boldsymbol{u})$ in (7) we formally obtain the Nitsche method of [9] *which can be interpreted as a stabilized multiplier method in the linear case* [15]. The augmented Lagrange multiplier method is, however, *not* a stabilized method and is subject to the standard problems of matching spaces for stability of the discrete problem, cf. [7].

3. Finite element methods

Assume that we are given a triangulation $\mathcal{T}^h$ of the domain Ω. We denote by h the meshsize of $\mathcal{T}^h$. We introduce the finite element space

$$\vec{V}^h=\{\boldsymbol{v} : \boldsymbol{v}\in\left[H^1(\Omega_i)\right]^d,\quad \boldsymbol{v}|_K\in[P^1(K)]^d,\ \forall K\in\mathcal{T}^h,\quad \boldsymbol{v}=\boldsymbol{0} \text{ on } \Gamma^{\mathrm{D}}\},$$

where $P^1(K)$ denotes the space of affine polynomials on K. On Γ^{C} we introduce a family of spaces Q^h of discrete multipliers. As a particular case, we will consider the spaces Q_i^h, $i=0$ or $i=1$, of piecewise constants or linears defined as follows: the interface Γ^{C} is decomposed as the union of the faces of the triangulation $\mathcal{T}^h$ on Γ^{C} which gives a set of faces $\mathcal{F}_{\mathrm{C}}^h$ consisting of the faces of simplices in $\mathcal{T}^h$.

Our particular choices of multiplier space are

$$Q_0^h=\{q\in Q : q|_K\in P^0(K),\ \forall K\in\mathcal{F}_{\mathrm{C}}^h\}, \tag{8}$$

and

$$Q_1^h=\{q\in C^0(\Gamma^{\mathrm{C}}) : q|_K\in P^1(K),\ \forall K\in\mathcal{F}_{\mathrm{C}}^h\}, \tag{9}$$

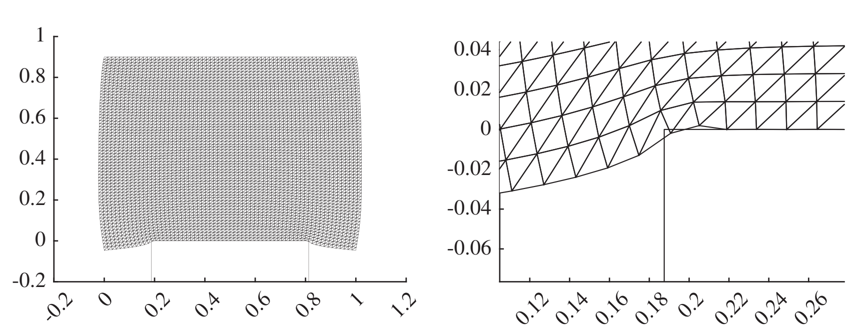

Fig. 1. Displacements using the unstabilized scheme.