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# Free vibration analysis for shells of revolution based on $p$-version mixed finite element formulation

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ABSTRACT

In this study, an effective $p$-version two-node mixed finite element is newly presented for predicting the free vibration frequencies and mode shapes of isotropic shells of revolution. The present element considering shear strains is based on Reissner–Mindlin shear deformation shell theory and Hellinger–Reissner variational principle. To improve the accuracy and resolve the numerical difficulties due to the spurious constraints, field-consistent stress parameters are employed corresponding to displacement shape functions with high-order hierarchical shape functions. The elimination of stress parameters and the reduction of the nodeless degrees by the Guyan reduction yield the standard stiffness and mass matrix. Results of the proposed element are compared with analytical, experimental and numerical solutions found in the literature. We can confirm a very satisfactory numerical behavior of the present $p$-version mixed element.



## 1. Introduction

Shells of revolution have been widely used in pressure vessels, cooling towers, bioengineering applications, aerospace structures and nuclear reactors, and span all branches of engineering because their double curvature has a significant effect on the behavior of a structure in carrying load. At the same time, it introduces complexity into the formulation and numerical calculation. Thus it has been a challenging subject to find satisfactory solutions for shells of revolution with any kind of boundary condition, as can be evidenced from a great number of literatures [1,2].

Besides analytical methods [3–5], many numerical studies such as the differential quadrature method [6,7], the boundary element method [8], the Ritz method[9], the strip method [10], and finite element methods [11–29] have been proposed. Among these different methods, the finite element method is recognized as the most versatile approach. Grafton and Strome [11] firstly proposed the element modeled as a straight-sided conical frustrum and extended to the case of non-axisymmetric loads. This harmonic axisymmetric shell element has one major advantage that the uncoupled circumferential harmonics make the analysis be performed independently for each harmonic.

The conventional low-order displacement-based finite element models have poor convergence and accuracy in predicting the stress field due to shear and membrane locking phenomena [12–15]. To overcome these problems, many techniques such as the selective/reduced integration finite element [16,17], the field-consistent element [15], the spline finite element [18–21], and the mixed finite element [22–24] have been proposed. The mixed finite elements based on multi-field variational principles can provide better convergence rates and accuracy in the stresses than the conventional displacement-based element. However, the mixed finite element having low-order interpolation functions in displacement field and inappropriate stress parameters still does not have sufficiently satisfactory numerical performance [22,23]. Another alternative strategy for avoiding locking phenomena is to use high-order $p$-version methods [25,26]. Chapell and Bathe [27] show that these $p$-version elements are free from locking in the displacement field and energy-norm for general shells, but not excused from locking in the stress field.

In this study, we propose a new two-node $p$-version mixed finite element for free vibration analysis of shells of revolution. The present element is based on the first-order Reissner–Mindlin shear deformation shell theory and the Hellinger–Reissner variational principle. The high-order nodeless hierarchical shape functions in the displacement field are introduced to enhance the accuracy in the prediction of higher vibration modes. The parameters for eight stress resultant variables, which are selected to be field-consistent to five displacement variables, can make possible to develop an element alleviating numerical difficulties such as shear and membrane locking, typical in the limits of shearless deformation and inextensional pure bending deformation. All field variables in the formulation are expanded into Fourier series with respect to the circumferential coordinate, and the

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initial two-dimensional problem is transformed into a series of independent one-dimensional problems. The stress parameters and the introduced nodelss degrees in the displacement field are sequentially removed in the final stages of element formulation through the functional stationarity and Guyan reduction, respectively.

The numerical performance of the present mixed element is evaluated in several examples. These numerical tests for comparison to available solutions in the literatures show that the present element has satisfactory convergence and accuracy in the free vibration analysis for shells of revolution.

## 2. Variational formulation

The $r$, $\theta$, and $z$-coordinate system in an element of a shell of revolution is shown in Fig. 1. The meridional radius of curvature is defined by $r_1 = ds/d\varphi$ and the arc length along the meridian is depicted by $s$. The positive normal direction $\zeta$ is outward while the center of curvature of the meridian is on the inside. The meridional, circumferential and normal displacements in the $s$, $\theta$ and $\zeta$ directions are expressed by $U$, $V$ and $W$, respectively.

By assuming that the meridional and circumferential displacements vary linearly across the shell thickness, while the normal displacement is constant, these displacements are represented in the following form of the generalized displacement components for an arbitrary point at location $(z,\ \theta,\ 0)$:

$$\mathbf{U} = \begin{Bmatrix} U(z,\theta,\zeta) \\ V(z,\theta,\zeta) \\ W(z,\theta,\zeta) \end{Bmatrix} = \begin{bmatrix} \zeta & 0 & 1 & 0 & 0 \\ 0 & \zeta & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \beta_s \\ \beta_\theta \\ u \\ v \\ w \end{Bmatrix} = \mathbf{TD} \quad (1)$$

where $u$, $v$, $w$ are the translation on the mid-surface along the meridional, circumferential, and normal directions, respectively, and $\beta_s$, $\beta_\theta$ are the rotations of the normal to the mid-surface along meridional and circumferential directions, respectively.

The relationship between strains and displacements in the interior of the domain can be stated as

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \varepsilon_{s\theta} \\ \kappa_s \\ \kappa_\theta \\ \kappa_{s\theta} \\ \gamma_s \\ \gamma_\theta \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial s} & 0 & \frac{1}{r_1} \\ 0 & 0 & \frac{\cos\phi}{r} & \frac{1}{r}\frac{\partial}{\partial\theta} & \frac{\sin\phi}{r} \\ 0 & 0 & \frac{1}{r}\frac{\partial}{\partial\theta} & \frac{\partial}{\partial s} & -\frac{\cos\phi}{r} \\ \frac{\partial}{\partial s} & 0 & 0 & 0 & 0 \\ \frac{\cos\phi}{r} & \frac{1}{r}\frac{\partial}{\partial\theta} & 0 & 0 & 0 \\ \frac{1}{2r}\frac{\partial}{\partial\theta} - \frac{\cos\phi}{2r} & \frac{1}{2}\frac{\partial}{\partial s} & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{r_1} & 0 & -\frac{\partial}{\partial s} \\ 0 & 1 & 0 & -\frac{\sin\phi}{r} & \frac{1}{r}\frac{\partial}{\partial\theta} \end{bmatrix} \begin{Bmatrix} \beta_s \\ \beta_\theta \\ u \\ v \\ w \end{Bmatrix} = \boldsymbol{\Delta}D \quad (2)$$

where $\varepsilon_s$, $\varepsilon_\theta$, $\varepsilon_{s\theta} = \varepsilon_{\theta s}$ are three in-plane strains; $\kappa_s$, $\kappa_\theta$, $\kappa_{s\theta} = \kappa_{\theta s}$ are two curvature changes and twisting strain; and $\gamma_s$, $\gamma_\theta$ are two transverse shear strains. In the present formulation, symmetric mixed strain components are preserved as approximately equivalent [6]. The relationship between stress resultants shown in Fig. 2 and strains for linearly elastic isotropic material are defined by

$$\boldsymbol{\sigma} = \begin{Bmatrix} N_s \\ N_\theta \\ N_{s\theta} \\ M_s \\ M_\theta \\ M_{s\theta} \\ Q_s \\ Q_\theta \end{Bmatrix} = \begin{bmatrix} E_1 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ E_2 & E_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_4 & E_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_5 & E_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_7 \end{bmatrix} \begin{Bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \varepsilon_{s\theta} \\ \kappa_s \\ \kappa_\theta \\ \kappa_{s\theta} \\ \gamma_s \\ \gamma_\theta \end{Bmatrix} = E\boldsymbol{\varepsilon} = \mathbf{C}^{-1}\boldsymbol{\varepsilon} \quad (3)$$

where $E$, $\nu$ and $t$ are the elastic modulus, Poisson's ratio and shell thickness, respectively, and $E_1 = Et/[2(1-\nu^2)]$, $E_2 = \nu E_1$, $E_3 = (1-\nu)E_1/2$, $E_4 = Et^3/[12(1-\nu)]$, $E_5 = \nu E_4$, $E_6 = (1-\nu)E_4/2$, $E_7 = k_s E_3$. The transverse shear factor is denoted by $k_s$.

Assuming a time-harmonic motion with the natural angular frequency $\omega$, Hamilton's principle for shells of revolution [30] can be written as

$$\delta\Pi_R = \delta\left\{ \int_0^{2\pi} \left[ \int_{s_1}^{s_2} L_R\, ds - \frac{1}{2}\rho\omega^2 \int_{s_1}^{s_2} \int_{-t/2}^{t/2} \mathbf{U}^T\mathbf{U}\, d\zeta r\, ds \right] d\theta \right\} = 0 \quad (4)$$

where $\rho$ is the mass density, and the energy density $L_R$ is depicted as

$$L_R = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon} - \int \boldsymbol{\varepsilon}^T\, d\boldsymbol{\sigma} = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon} - \tfrac{1}{2}\boldsymbol{\sigma}^T\mathbf{C}^T\boldsymbol{\sigma} \quad (5)$$

The vectors of stress resultants, strains and displacements vectors in Eqs. (1) and (4) are written in terms of a partial harmonic Fourier series expansion with respect to the circumferential coordinate $\theta$ as

$$\boldsymbol{\sigma} = \begin{Bmatrix} N_s \\ N_\theta \\ N_{s\theta} \\ M_s \\ M_\theta \\ M_{s\theta} \\ Q_s \\ Q_\theta \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} N_s^{(n)}\ \cos\ n\theta \\ N_\theta^{(n)}\ \cos\ n\theta \\ N_{s\theta}^{(n)}\ \sin\ n\theta \\ M_s^{(n)}\ \cos\ n\theta \\ M_\theta^{(n)}\ \cos\ n\theta \\ M_{s\theta}^{(n)}\ \sin\ n\theta \\ Q_s^{(n)}\ \cos\ n\theta \\ Q_\theta^{(n)}\ \sin\ n\theta \end{Bmatrix} \quad (6a)$$

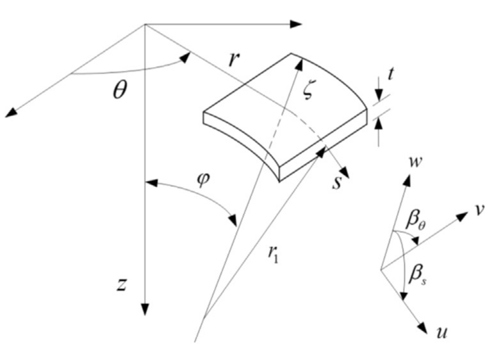


**Fig. 1.** The element of a shell of revolution and the coordinate system.

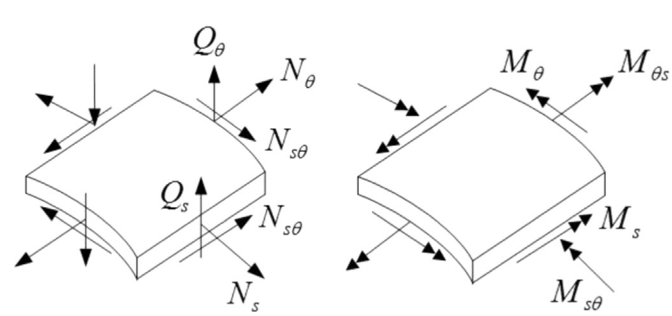


**Fig. 2.** Force and moment resultants acting on a shell element.