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On numerical solution of elastic–plastic problems by using configurational force driven adaptive methods

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ABSTRACT

In this paper, the concept of configurational forces is introduced in the context of finite element mesh refinement for elastic–ideally plastic problems. This paper also includes the numerical computation of configurational forces in the elastic and plastic domains. Methods are demonstrated on three plane problems where the analytical solution is available. The first example is a thick-walled tube loaded by internal pressure. This simple, one dimensional problem allows computation of configurational volume forces analytically to validate the finite element (FE) results. The second example is Galin's problem that involves an infinite plate with a circular hole loaded by biaxial tension at the infinity. This is a two dimensional problem for which the analytical solution is known with some restrictions for elastic–ideally plastic case when Tresca yield criterion is considered. The last example introduces another plane problem that follows Naghdi's solution on infinite wedges. For this, a new analytical solution is presented for plane stress state using von Mises yield criterion with a uniform shear loading along the boundary. R-, h-, and combined adaptive procedures are demonstrated on the above examples. Since exact stress fields are known, the norm of the difference between numerical and analytical solutions is used as the measure of error.

1. Introduction

Configurational mechanics, presented by Eshelby in the early fifties, is widely used in several areas of theoretical continuum mechanics and computational mechanics (see [8,10]). The configurational force (or often called material force) describes the change in energy of a given system with respect to the configuration. This means that the energy change caused by the reposition of an inhomogeneity inside the material could be considered as a force on it. The configurational force method is widely used in context with finite elements. One of these applications is the configurational force driven finite element mesh refinement. The so-called r-adaptive FE mesh refinement strategy was introduced by Braun [2] followed by several papers using the concept of configurational forces to drive h- and rh-adaptive procedures for purely elastic problems [13,15,23]. Braun's method is based on the property that the configurational force [10] must be zero in the interior of the homogeneous material [13]. However, finite element computations do not fulfill this requirement and nodal configurational forces appear on interior nodes too. These vectors are the gradient of the total potential of the system indicating the dependence of the solution on the initial nodal configuration. When an optimal mesh is used, internal configurational forces vanish and the total potential of the structure is minimal. Since these error configurational force vectors point in the direction of increasing total potential, moving the nodes in the opposite direction yields the optimal mesh structure. These methods were demonstrated in several articles for the elastic case, e.g. [2,9,13,14]. Some numerical shape and structural optimization techniques are also based on this theory. Furthermore, as the configurational force indicates the error on finite element meshes, computation of this quantity is also suitable to drive h-adaptive mesh refinement [13] and there are a few examples in the literature for the application of hp-refinement strategy [23]. Most of these papers are dealing with elastic deformation but configurational forces can also be computed in elastic–plastic case for small strains [20] or even in the case of a large strain formulation [17,18,21]. Examples for elastic–plastic problems can also be found in [24]. Remarkable practical applications of configurational forces appear in the field of non-linear fracture mechanics by the numerical computation of the J-integral, as can be seen in papers of Nguyen et al. [22] and Simha et al.[24].

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This paper shows that configurational force based adaptive mesh refinement strategies are also applicable in the case of elastic–ideally plastic deformation. The material equilibrium equation that must be fulfilled to reach optimal mesh configuration is different from the elastic case. The plastic part of configurational forces becomes zero in the purely elastic domain. In the following, small strain is considered. For validation of the finite element results a few examples, which have analytical solution, are also presented. These kind of solutions are very limited in case of elastic–plastic problems. The first test problem is a thick-walled cylinder subjected to internal pressure. A closed-form solution to this problem has been derived by Hill [6], and it can also be found in the literature, e.g., Chakrabarty [3], Lubliner [12]. In this example, we derive an analytical expression for the volume configurational force to validate the proposed algorithm. The second example analyses an infinite plate with a circular hole loaded by biaxial tension. The material is modelled as elastic–perfectly plastic with the Tresca yield criterion. This problem has an analytical solution with certain limitation presented by Galin [5]. The third example is an infinite elastic–plastic wedge, loaded by a uniformly distributed shear on its edge. The wedge problem was investigated by Naghdi et al. [19] subjected to different loading conditions and yield criteria.

In this contribution, a novel solution is proposed that is new to the best of our knowledge. In the wedge problem, the stress state depends on a single variable in cylindrical coordinate system. However, the presence of a singular point at the corner of the wedge makes this example excellent to demonstrate adaptive methods. The solution for the stress field is given in the Appendix. This paper is organized as follows. The theoretical background of computing the configurational forces is presented in the second section. The third section briefly summarizes the finite element computation technique of nodal error configurational forces. In the fourth chapter analytical volume configurational forces are presented through the test problems mentioned above. On these examples the configurational force driven r-, h- and rh-adaptive methods are demonstrated.

Notations: $\boldsymbol{\Sigma}, \boldsymbol{\sigma}$ — second order tensors, $\boldsymbol{G}, \boldsymbol{g}$—vector quantities, div(∗)—the divergence of ∗, $\boldsymbol{\delta}$—second order identity tensor, **grad**(∗)—gradient operator, $a, b, c, \alpha, \beta, \gamma$—scalar quantities , $\bar{\bullet}$—complex conjugate of a given quantity. The superscripts T and -1 denote transpose and inverse, and the following symbolic operations apply: $\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$, $(\mathbf{A} \cdot \boldsymbol{b})_i = A_{ij} b_j$, $\mathbf{A} : \mathbf{B} = A_{ij} B_{ij}$.

2. Theoretical background

In this section, the governing equations of the configurational force are recapitulated for linear elastic and elastic–perfectly plastic material models at small deformation. This outline follows the treatments of Gross et al. [7], Müller et al. [16], and Simha et al. [24].

Elastic domain: The concept of the configurational force can be described as an energy change in the system considered with respect to its configuration (e.g. Kienzler and Herrmann [10]). Therefore, the gradient of strain energy density function for linear elastic, isotropic and inhomogeneous material can be expressed as

$$\frac{\partial W}{\partial \boldsymbol{x}} = \frac{\partial W}{\partial \boldsymbol{\epsilon}} : \frac{\partial \boldsymbol{\epsilon}}{\partial \boldsymbol{x}} + \left.\frac{\partial W}{\partial \boldsymbol{x}}\right|_{explicit}, \tag{1}$$

where $\boldsymbol{\epsilon}$ is the linear strain tensor, and the subscript *explicit* denotes the explicit derivative of W with respect to the position vector $\boldsymbol{x}$.

The elastic constitutive law is defined by

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\epsilon}}, \tag{2}$$

where $\boldsymbol{\sigma}$ is the symmetric stress tensor.

By using this relationship, the Eq. (1) takes the form

$$\frac{\partial W}{\partial \boldsymbol{x}} = \boldsymbol{\sigma} : \mathbf{grad}\, \boldsymbol{\epsilon} + \left.\frac{\partial W}{\partial \boldsymbol{x}}\right|_{explicit} \equiv \boldsymbol{\sigma} : (\mathbf{grad\, grad}\, \boldsymbol{u}) + \left.\frac{\partial W}{\partial \boldsymbol{x}}\right|_{explicit}, \tag{3}$$

where $\boldsymbol{u}$ is the displacement field.

The equilibrium equation can be written as

$$\mathrm{div}\, \boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0}, \tag{4}$$

where $\boldsymbol{f}$ denotes the physical body (or volume) force.

Then, combining (3) and (4), and using the following two relationships

$$\mathrm{div}(\mathbf{grad}^T \boldsymbol{u} \cdot \boldsymbol{\sigma}) = \boldsymbol{\sigma} : (\mathbf{grad\, grad}\, \boldsymbol{u}) + \mathbf{grad}^T \boldsymbol{u} \cdot \mathrm{div}\, \boldsymbol{\sigma}, \tag{5}$$

and

$$\frac{\partial W}{\partial \boldsymbol{x}} \equiv \mathbf{grad}\, W = \mathrm{div}(W\boldsymbol{\delta}), \tag{6}$$

we obtain

$$\mathrm{div}\underbrace{(W\boldsymbol{\delta} - \mathbf{grad}^T \boldsymbol{u} \cdot \boldsymbol{\sigma})}_{\boldsymbol{\Sigma}} + \left(\underbrace{-\mathbf{grad}^T \boldsymbol{u} \cdot \boldsymbol{f} - \left.\frac{\partial W}{\partial \boldsymbol{x}}\right|_{explicit}}_{\boldsymbol{g}} \right) = \mathbf{0}, \tag{7}$$

where $\boldsymbol{\Sigma}$ is the Eshelby stress tensor and $\boldsymbol{g}$ is the configurational body force. If the physical body force, $\boldsymbol{f}$ is zero, and a homogeneous body is considered, the vector $\boldsymbol{g}$ becomes zero [24].

Finally, we assume that $\mathbf{g} = \mathbf{0}$, and the function W is defined by

$$W = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\epsilon}. \tag{8}$$

Then, the configurational force can be obtained by integrating the first term given in Eq. (7) over the elastic Ω_e domain

$$\boldsymbol{G} = \int_{\Omega_e} \mathrm{div}\left[\frac{1}{2}(\boldsymbol{\sigma} : \boldsymbol{\epsilon})\boldsymbol{\delta} - \mathbf{grad}^T \boldsymbol{u} \cdot \boldsymbol{\sigma}\right] dV = 0. \tag{9}$$

Elastic–plastic domain: In the classical theory of plasticity, the total strain is assumed to be the sum of the elastic and plastic strain

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p, \tag{10}$$

and, in the case of elastic–perfect plastic materials without hardening, the strain energy density function, W depends only on the elastic strain (see e.g. [16,24]), namely

$$W = W_e(\boldsymbol{\epsilon}^e). \tag{11}$$

Recall the elastic constitutive relation given by the expression

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\epsilon}^e}. \tag{12}$$

Then, the gradient of W, using (10) is given by

$$\frac{\partial W}{\partial \boldsymbol{x}} = \frac{\partial W}{\partial \boldsymbol{\epsilon}^e} : \frac{\partial \boldsymbol{\epsilon}^e}{\partial \boldsymbol{x}} = \boldsymbol{\sigma} : \mathbf{grad}\, \boldsymbol{\epsilon}^e = \boldsymbol{\sigma} : \mathbf{grad}\, \boldsymbol{\epsilon} - \boldsymbol{\sigma} : \mathbf{grad}\, \boldsymbol{\epsilon}^p$$
$$= \boldsymbol{\sigma} : (\mathbf{grad\, grad}\, \boldsymbol{u}) - \boldsymbol{\sigma} : \mathbf{grad}\, \boldsymbol{\epsilon}^p. \tag{13}$$

Note that the quantity associated with the explicite derivative of W with respect to the position vector $\boldsymbol{x}$ is also omitted (homogeneous material).

Eq. (13), using (4)–(6), can be rewritten in the following form

$$\mathrm{div}\left[\frac{1}{2}(\boldsymbol{\sigma} : \boldsymbol{\epsilon}^e)\boldsymbol{\delta} - \mathbf{grad}^T \boldsymbol{u} \cdot \boldsymbol{\sigma}\right] + \boldsymbol{\sigma} : \mathbf{grad}\, \boldsymbol{\epsilon}^p = 0. \tag{14}$$

It is noted here that the strain energy density function is defined by $W = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\epsilon}^e$, and the body force $\boldsymbol{f} = \mathbf{0}$ as before.