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A globally conforming method for solving flow in discrete fracture networks using the Virtual Element Method

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ABSTRACT

A new approach for numerically solving flow in Discrete Fracture Networks (DFN) is developed in this work by means of the Virtual Element Method (VEM). Taking advantage of the features of the VEM, we obtain global conformity of all fracture meshes while preserving a fracture-independent meshing process. This new approach is based on a generalization of globally conforming Finite Elements for polygonal meshes that avoids complications arising from the meshing process. The approach is robust enough to treat many DFNs with a large number of fractures with arbitrary positions and orientations, as shown by the simulations. Higher order Virtual Element spaces are also included in the implementation with the corresponding convergence results and accuracy aspects.

1. Introduction

The present work deals with a new approach based on the Virtual Element Method (VEM) for the simulation of the flow in Discrete Fracture Networks (DFNs). DFN models are one of the possible approaches for simulating subsurface flows and they consist of a set of planar polygons in 3D space resembling the fractures in the underground. Each fracture is modelled individually, as opposed to continuum models with equivalent porosity, and, for geological formations with a sparse fracture network that mainly affects the flow path, this approach is recommended [1,2]. DFNs are used in a wide range of applications such as pollutant percolation, gas recovery, aquifers and reservoir analysis [3,4].

Stationary flow in a DFN is modelled using Darcy's law and introducing a transmissivity tensor for each fracture that depends on its aperture and its resistance to flow. The surrounding rock matrix is considered impervious. The goal is to obtain the hydraulic head distribution in the system, which is the sum of the pressure head and the elevation. Fluid can only flow through fractures and across intersections between fractures, also called traces, but no tangential flow is considered along traces. The hydraulic head is a continuous function, but with discontinuous derivatives across the traces, which act as sources/sinks of flow. More complex models for the flow in the fractures can be found in the literature [5]. Since little is known about the subsurface fractures, stochastic models are used in order to determine distributions of aperture, hydrological properties, size, orientation, density, and aspect ratio of the fractures.

Geometrical complexity is the greatest challenge when dealing with DFN-based simulations. Since the fracture generation has a random component, many complex situations arise that render the meshing process very complicated and sometimes impossible, e.g. very small angles, very close and almost parallel traces, high disparity of traces lengths, etc. In order to use traditional finite elements, fracture grids have to match in all the intersections between fractures, since these are discontinuity interfaces for the first order derivatives of the solution. All the aforementioned geometrical configurations complicate the meshing process and are the biggest obstacle in the discretization of the problem because it becomes very computationally demanding to obtain a good mesh from such a badly predisposed geometry. Furthermore, the meshing procedure depends on the whole DFN and is not independent for each fracture. When a large DFN is considered that can have thousands of fractures, mesh conformity requirements can lead to a very high number of elements that are far more than those demanded by the required level of accuracy. In [6], a BEM (Boundary Element Method) was applied that aims to minimize core memory usage by defining and storing only a relation between nodal fluxes and hydraulic head on traces for each fracture. The problem of obtaining a good globally conforming mesh is the subject of ongoing research. In [7], an adaptive mesh refinement method is described that aims for a high resolving mesh. Previous works [8,9] suggest a simplification of the geometry to ease meshing. Monodimensional pipes joining fractures, instead of traces, have been put forward as an alternative in

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[10,11]. In [12], a mixed formulation and a mesh modifying procedure were used to solve DFNs and reducing the number of elements for each fracture. Another mixed formulation was used in [13], where local corrections of traces are applied in order to obtain a globally conforming mesh. The mortar method was used to impose conditions between fractures with non-matching grids to obtain a mixed hybrid formulation in [14], with a subsequent generalization in [15] that includes trace intersections within a fracture. A novel approach was proposed in [16–19] in which the problem was reformulated as a PDE-constrained optimization. The minimization of a properly defined functional is adopted to enforce hydraulic head continuity and flux conservation at fracture intersections. Traditional finite elements (FEM) as well as extended finite elements (XFEM) were implemented to solve the problem.

In this work, we aim to provide an easy, natural way for generating conforming meshes for complex DFN problems using the VEM. The proposed approach is a generalization of traditional conforming finite elements, keeping the method as simple and streamlined as possible. Some of the ideas presented here were present in a previous work by the authors [20], that introduced Virtual Elements (VEM) to DFNs. In [20] the VEM is used on locally conforming meshes and an optimization approach is adopted to handle the non-conformity of the global mesh. Here both local and global conformity are enforced, and classical approaches, borrowed from the domain decomposition methods, can be used to solve the problem. We make absolutely no assumptions on the meshing procedure, which is done independently for each fracture and without any consideration of the position of the traces. Traces are not modified in any way, and using some of the features of the VEM, local and global conformity for the mesh is obtained by means of splitting the original elements of the meshes independently generated on each fracture into polygons of an arbitrary number of vertices.

Using Lagrange multipliers we obtain a hybrid system that can be solved with different methods, including FETI algorithms for domain decomposition.

Section 2 provides the formulation of the DFN problem in the present context, whereas a brief summary of the VEM is reported in Section 3, and in Section 4 the proposed method is described in detail. Numerical results are presented in Section 5, where some convergence results are given and the applicability of the method to DFNs is discussed.

2. The continuous problem

Let us consider a set of open convex planar polygonal fractures $F_i \subset \mathbb{R}^3$ with $i=1,\dots,N$, with boundary ∂F. Our DFN is $\Omega = \bigcup_i F_i$, with boundary $\partial\Omega$. Even though the fractures are planar, their orientations in space are arbitrary, such that Ω is a 3D set. The set $\Gamma_D \subset \partial\Omega$ is where Dirichlet boundary conditions are imposed, and we assume $\Gamma_D \neq \emptyset$, whereas $\Gamma_N = \partial\Omega \setminus \Gamma_D$, is the portion of the boundary with Neumann boundary conditions. Dirichlet and Neumann boundary conditions are prescribed by the functions $h^D \in H^{1/2}(\Gamma_D)$ and $g^N \in H^{-1/2}(\Gamma_N)$ on the Dirichlet and Neumann part of the boundary, respectively. We further set $\Gamma_{iD} = \Gamma_D \cap \partial F_i$, $\Gamma_{iN} = \Gamma_N \cap \partial F_i$, and $h_i^D = h^D_{|\Gamma_{iD}}$ and $g_i^N = g^N_{|\Gamma_{iN}}$. The set $\mathcal{T}$ collects all the traces, i.e. the intersections between fractures, and each trace $T \in \mathcal{T}$ is given by the intersection of exactly two fractures, $T = \overline{F}_i \cap \overline{F}_j$, such that there is a one to one relationship between a trace T and a couple of fracture indexes $\{i,j\} = \mathcal{I}(T)$. We will also denote by $\mathcal{T}_i$ the set of traces belonging to fracture F_i.

Subsurface flow is governed by the gradient of the hydraulic head $H = \mathcal{P} + \zeta$, where $\mathcal{P} = p/(\varrho g)$ is the pressure head, p is the fluid pressure, g is the gravitational acceleration constant, ϱ is the fluid density and ζ is the elevation.

We define the following functional spaces:

$$V_i = H_0^1(F_i) = \left\{ v \in H^1(F_i) : v_{|\Gamma_{iD}} = 0 \right\},$$

$$V_i^D = H_D^1 F_i = \left\{ v \in H^1(F_i) : v_{|\Gamma_{iD}} = h_i^D \right\},$$

and

$$V = \left\{ v : v_{|F_i} \in V_i, \;\; \forall i = 1,\dots,N, \;\; \gamma_T(v_{|F_i}) = \gamma_T(v_{|F_j}), \;\; \forall T \in \mathcal{T}_i, \;\; \{i,j\} = \mathcal{I}(T) \right\},$$

where γ_T is the trace operator onto T. It is then possible to formulate the DFN problem, given by the Darcy's law in its weak form on the fractures with additional constraints of continuity of the hydraulic head across the traces: for $i=1,\dots,N$, find $H_i \in V_i^D$ such that $\forall v \in V$

$$\sum_{i=1}^{N} \int_{F_i} \mathcal{K}_i \nabla H_i \nabla v_{|F_i} \, \mathrm{d}F_i = \sum_{i=1}^{N} \left(\int_{F_i} f_i v_{|F_i} \, \mathrm{d}F_i + \langle g_i^N, v_{|\Gamma_{N_i}} \rangle_{H^{-\frac{1}{2}}(\Gamma_{N_i}), H^{\frac{1}{2}}(\Gamma_{N_i})} \right),$$

$$\gamma_T(H_i) = \gamma_T(H_j), \quad \forall T \in \mathcal{T}, \quad \{i,j\} = \mathcal{I}(T)$$

where $\mathcal{K}_i$ is the fracture transmissivity tensor, that we assume is constant on each fracture. The second equation represents the continuity of the hydraulic head across traces. On each fracture of the DFN the following bilinear form $a_i : V_i \times V_i \mapsto \mathbb{R}$ is defined as

$$a_i(H_i, v_{|F_i}) = \int_{F_i} \mathcal{K}_i \nabla H_i \nabla v_{|F_i} \, \mathrm{d}F_i. \tag{2.1}$$

3. The Virtual Element Method

This section provides a quick overview of the VEM, recalling the main features useful in the present context. We refer the reader to the original paper [21] for a proper introduction and to [22] for a guide on implementation. Further developments can be found in [23–26]. The VEM has also been applied to problems in elasticity [27], plate bending [28], the Stokes problem [29] and has sparked interest in other applications as well.

Borrowing ideas from the Mimetic Finite Difference method [30,31], the VEM can be regarded as a generalization of regular finite elements to meshes made up by polygonal elements of any number of edges. The discrete functional space on each element has, in general, not only polynomial functions but also other functions that are only known at a certain set of degrees of freedom. Given a bilinear form to be approximated with the VEM, our goal is to build a discrete bilinear form that coincides with the exact one when at least one of the arguments is a polynomial. For the other cases, a rough approximation that scales in a desired way is enough to obtain the desired convergence qualities of the method.

Given a domain $F \subset \mathbb{R}^2$, a mesh τ_h on F, made of polygons $\{E\}$ with mesh parameter h (i.e. the square root of the maximum element area), and the space of the polynomials of maximum order k, $\mathcal{P}_k$, let us define the local space $V_{k,h}^E$ for a given polynomial degree k as

$$V_{k,h}^E = \left\{ v_h \in H^1(E) : v_{h|\partial E} \in C^0(\partial E), \;\; v_{h|e} \in \mathcal{P}_k(e) \;\; \forall e \subset \partial E, \;\; \Delta v_h \in \mathcal{P}_{k-2}(E) \right\}$$

where ∂E is the border of E, and e an edge.

From the above definition it is clear that the space $\mathcal{P}_k(E)$ is a subset of $V_{k,h}^E$. We define the following degrees of freedom for each element E:

- The value of v_h at the vertices of E;
- The value of v_h at $k-1$ internal points on each edge of E;