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Direct-differentiation-based sensitivity analysis of an axisymmetric finite element formulation including torsion

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ABSTRACT

The shape sensitivity analysis using the direct differentiation method is developed for the case of axisymmetry with superposed torsion in the framework of finite strains and elastic material behavior. The accuracy of this method is compared to results obtained by the finite difference method in two simple examples.

1. Introduction

In case of axisymmetric or near-axisymmetric geometries, generalized axisymmetry offers the possibility to reduce a three-dimensional model to a two-dimensional model with additional degrees of freedom in tangential direction. Generally speaking, the geometry is defined in a plane, and a discrete Fourier expansion is operated in tangential direction, in order to take into account either a non-axisymmetric geometry or, as in most cases, non-axisymmetric loads. The introduction of a single degree of freedom in tangential direction enables the treatment of axisymmetry with superposed torsion. The generalized axisymmetry thus offers the possibility to strongly reduce the computational effort by reducing the number of degrees of freedom in comparison to a full three-dimensional model. This may especially be of interest in optimization or inverse problems, where the same finite element model has to be recomputed a significant number of times. An additional advantage of an axisymmetric model concerns contact mechanics, as the treatment of contact surfaces in three dimensions is reduced to that of contact curves in two dimensions, which greatly simplifies modeling and enables the use of methods for more efficient contact treatment, like for example contact surface discretizations with C^1-continuity [1].

Among the first authors to produce a finite element formulation for generalized axisymmetry are [2,3] for linear elasticity and [4] for inelastic materials, both at small strains. Finite strain formulations for generalized axisymmetry in solid mechanics have been presented in [5], and later on in various contexts like thermomechanical homogenization [6], mixed or enhanced strain formulations [7–9], ALE formulations [6], consolidation analysis [10], or others [11,12]. Contact mechanics in the context of generalized axisymmetry has been treated in [14], for example. One of the major challenges in finite strains is that the base vectors vary with position [13]. The treatment of generalized axisymmetry may be executed either in mathematical or physical coordinates [15], using different ways to arrive at the equations of virtual work. Different mathematical concepts are use for an efficient treatment of generalized axisymmetry. For example, use is made of a mathematical shifter tensor in [16] for deriving the kinematics of generalized axisymmetry. In [5], the finite element equations use cylindrical coordinates for the description of the geometry, but the displacements are kept in Cartesian coordinates, in order to avoid displacement locking. Issues related to the numerical integration of axisymmetric finite element formulations are discussed in [17], an improved treatment of incompressibility is detailed in [18] and methods for reducing mesh locking in the context of axisymmetry are given in [19–21], for example.

Sensitivity analysis, i.e. the calculation of derivatives with respect to model parameters like material or shape parameters, amongst others, is useful in gradient-based solution methods of inverse problems or shape optimization. Various aspects of sensitivity analysis in standard axisymmetry have been presented in [22–24], for example. Recently, the sensitivity analysis using the adjoint state method for generalized axisymmetry with fully asymmetric loading has been presented in [25] for the case of small strains. In the present work, sensitivity analysis based on the direct differentiation method is presented for finite strains for the most simple case of generalized axisymmetry, i.e. axisymmetry with superposed torsion. First, the kinematics of

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generalized axisymmetry are outlined. Then, the linearization and sensitivity analysis of the kinematic quantities are presented and introduced into the virtual work equation. With the sensitivity of the Cauchy stress, the discretized formulation of the linearized virtual work equation is given. Finally, the performance of the sensitivity analysis is assessed in two examples by comparing the sensitivities obtained by direct differentiation to those obtained by the finite difference method.

2. Generalized axisymmetry

2.1. Kinematics

According to [13], the geometry $\mathbf{x}$ in the current state, described by the coordinates r, z and ϕ,

$$\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z, \tag{1}$$

is related to the initial geometry with coordinates R, Z and Φ through the radial displacement u, the vertical displacement v and the angular displacement θ:

$$\begin{aligned} r &= R + u(R,Z) \\ z &= Z + v(R,Z) \\ \phi &= \Phi + \theta(R,Z). \end{aligned} \tag{2}$$

The current basis vectors are $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$, the first two depending on ϕ. Then, the deformation gradient can be expressed as

$$\mathbf{F} = \begin{pmatrix} \frac{\partial r}{\partial R} & \frac{\partial r}{\partial Z} & 0 \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial Z} & 0 \\ r\frac{\partial \phi}{\partial R} & r\frac{\partial \phi}{\partial Z} & \frac{r}{R}\frac{\partial \phi}{\partial \Phi} \end{pmatrix} = \begin{pmatrix} 1+\frac{\partial u}{\partial R} & \frac{\partial u}{\partial Z} & 0 \\ \frac{\partial v}{\partial R} & 1+\frac{\partial v}{\partial Z} & 0 \\ r\frac{\partial \theta}{\partial R} & r\frac{\partial \theta}{\partial Z} & 1+\frac{u}{R} \end{pmatrix}. \tag{3}$$

2.2. Variational calculus

Taking into account that in polar or cylindrical coordinates, the non-zero variations of the basis vectors are

$$\begin{aligned} \delta\mathbf{e}_r &= \delta\theta\, \mathbf{e}_\phi \\ \delta\mathbf{e}_\phi &= -\delta\theta\, \mathbf{e}_r, \end{aligned} \tag{4}$$

the variation of the deformation gradient, Eq. (3), becomes

$$\delta\mathbf{F} = \begin{pmatrix} \frac{\partial \delta u}{\partial R} & \frac{\partial \delta u}{\partial Z} & 0 \\ \frac{\partial \delta v}{\partial R} & \frac{\partial \delta v}{\partial Z} & 0 \\ r\frac{\partial \delta\theta}{\partial R} & r\frac{\partial \delta\theta}{\partial Z} & \frac{\delta u}{R} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta u\frac{\partial \phi}{\partial R} & \delta u\frac{\partial \phi}{\partial Z} & 0 \end{pmatrix} + \delta\theta \begin{pmatrix} -r\frac{\partial \phi}{\partial R} & -r\frac{\partial \phi}{\partial Z} & -\frac{r}{R} \\ 0 & 0 & 0 \\ \frac{\partial r}{\partial R} & \frac{\partial r}{\partial Z} & 0 \end{pmatrix}. \tag{5}$$

With the partial velocity gradient

$$\delta\tilde{\mathbf{L}} = \begin{pmatrix} \frac{\partial \delta u}{\partial r} & \frac{\partial \delta u}{\partial z} & 0 \\ \frac{\partial \delta v}{\partial r} & \frac{\partial \delta v}{\partial z} & 0 \\ r\frac{\partial \delta\theta}{\partial r} & r\frac{\partial \delta\theta}{\partial z} & \frac{\delta u}{r} \end{pmatrix} \tag{6}$$

and the sparse anti-symmetric tensor

$$\delta\boldsymbol{\Xi} = \begin{pmatrix} 0 & 0 & -\delta\theta \\ 0 & 0 & 0 \\ \delta\theta & 0 & 0 \end{pmatrix}, \tag{7}$$

the variation of the deformation gradient can be expressed as

$$\delta\mathbf{F} = \delta\tilde{\mathbf{L}}\mathbf{F} + \delta\boldsymbol{\Xi}\,\mathbf{F}, \tag{8}$$

where the last term in Eq. (5), which stems from the basis variation, equals the last term in Eq. (8). The variation of the velocity gradient $\delta\mathbf{L}$ is subsequently obtained by multiplying Eq. (8) by the inverse of the deformation gradient, yielding

$$\delta\mathbf{L} = \delta\mathbf{F}\,\mathbf{F}^{-1} = \delta\tilde{\mathbf{L}} + \delta\boldsymbol{\Xi}. \tag{9}$$

2.3. Linearization and shape sensitivity

In opposition to sensitivities of kinematic entities with respect to material parameters, where the sensitivities correspond to a linearization, shape sensitivities include additional terms because the initial and current coordinates include the shape change. Thus, for shape sensitivities, Eq. (2) has to be rewritten using a reference geometry, $\mathbf{X}^r$, a design velocity $\mathbf{V}$ and a parameter τ quantifying the design change, as described in [26,27]. In the following, the sensitivity of a generic variable α, designating either a scalar or a first or second order tensor, is defined as

$$[\alpha]^\circ = \lim_{\tau\to 0}\frac{\alpha(\mathbf{X}_r + \tau\mathbf{V}(\mathbf{X}_r)) - \alpha(\mathbf{X}_r)}{\tau}. \tag{10}$$

It should be noted that the design change is prescribed on the boundary of the solid, but the way the resulting design velocity propagates into the interior of the solid is determined using methods like the finite difference method, isoparametric mapping, boundary displacement method or fictitious load method [26,27]. In the present work, the boundary displacement method is used. In generalized axisymmetry, the shape change only affects the in-plane initial coordinates, i.e. no shape change in tangential direction, or shape change dependency in tangential direction, takes place. Thus one can extend Eq. (2) into

$$\begin{aligned} r &= R^r + V_r\tau + u(R,Z) \\ z &= Z^r + V_z\tau + v(R,Z) \\ \phi &= \Phi + \theta(R,Z). \end{aligned} \tag{11}$$

The linearization of the deformation gradient gives a similar result to Eq. (5), notably

$$[\mathbf{F}]^\circ = \left[\tilde{\mathbf{L}}\right]^\circ \mathbf{F} + [\boldsymbol{\Xi}]^\circ\,\mathbf{F} - \mathbf{F}\frac{\partial\mathbf{V}}{\partial\mathbf{X}}, \tag{12}$$

with

$$\frac{\partial\mathbf{V}}{\partial\mathbf{X}} = \begin{pmatrix} \frac{\partial V_r}{\partial R} & \frac{\partial V_r}{\partial Z} & 0 \\ \frac{\partial V_z}{\partial R} & \frac{\partial V_z}{\partial Z} & 0 \\ 0 & 0 & \frac{V_r}{R} \end{pmatrix}. \tag{13}$$

The last term in Eq. (12) comes from the dependency of the initial geometry on the shape velocity in the differentiation with respect to the initial geometry, as explained in [27]. Multiplying Eq. (12) with the inverse of the deformation gradient, one gets

$$[\mathbf{L}]^\circ = [\mathbf{F}]^\circ\,\mathbf{F}^{-1} = \left[\tilde{\mathbf{L}}\right]^\circ + [\boldsymbol{\Xi}]^\circ - \mathbf{F}\frac{\partial\mathbf{V}}{\partial\mathbf{x}}, \tag{14}$$

where

$$\frac{\partial\mathbf{V}}{\partial\mathbf{x}} = \frac{\partial\mathbf{V}}{\partial\mathbf{X}}\mathbf{F}^{-1} \tag{15}$$

and the tensors $\left[\tilde{\mathbf{L}}\right]^\circ$ and $[\boldsymbol{\Xi}]^\circ$ are defined as

$$\left[\tilde{\mathbf{L}}\right]^\circ = \begin{pmatrix} \frac{\partial[r]^\circ}{\partial r} & \frac{\partial[r]^\circ}{\partial z} & 0 \\ \frac{\partial[z]^\circ}{\partial r} & \frac{\partial[z]^\circ}{\partial z} & 0 \\ r\frac{\partial[\phi]^\circ}{\partial r} & r\frac{\partial[\phi]^\circ}{\partial z} & \frac{\delta u}{r} \end{pmatrix} \tag{16}$$

and

$$[\boldsymbol{\Xi}]^\circ = \begin{pmatrix} 0 & 0 & -[\phi]^\circ \\ 0 & 0 & 0 \\ [\phi]^\circ & 0 & 0 \end{pmatrix}. \tag{17}$$

Eqs. (16) and (17) are similar to the definitions of the respective variations in Eqs. (6) and (7), respectively, with the difference that the former use derivatives of sensitivities of the coordinates, whereas the latter use derivatives of the variations of the