Finite Elements in Analysis and Design 45 (2009) 395–405

Contents lists available at ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.com/locate/finel

Nonlinear finite element analysis of orthotropic and prestressed membrane structures

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ARTICLE INFO

Article history:
Received 13 May 2008
Accepted 21 November 2008
Available online 5 March 2009

Keywords:
Membrane structures
Orthotropy
Triangular finite elements
Prestressing
Principal fiber orientation

ABSTRACT

A new methodology for the geometrically nonlinear analysis of orthotropic membrane structures using triangular finite elements is presented. The approach is based on writing the constitutive equations in the principal fiber orientation of the material. A direct consequence of the fiber orientation strategy is the possibility to analyze initially out-of-plane prestressed membrane structures. An algorithm to model wrinkling behavior is also described. Examples of application to a number of membrane structures are presented.

1. Introduction

Membrane structures are used for many purposes in engineering and architecture. They are typically built with very light materials which are optimally used. These structures are characterized because they are only subjected to in-plane axial forces. Examples include textile covers and roofs, aircraft and space structures, parachutes, automobile airbags, sails, windmills, human tissues and long span structures.

A membrane is essentially a thin shell with no flexural stiffness. Consequently a membrane cannot resist any compression at all. However, membrane theory accounts for tension and compression stresses, and the need for a computational procedure that takes into account tension stresses only is needed. This effect is modeled in this work with a wrinkling algorithm. In membrane theory only the in-plane stress resultants are taken into account. The position of points on the two-dimensional surface in the Euclidean space gives the deformation state for a membrane. A numerical solution for membranes may be found using the finite element method. Finite element analysis of membrane structures for small deformations can be found in Zienkiewicz and Taylor [1], Cook et al. [2] or Oñate [3]. Theory for large deformations of thin membranes and shells have been proposed by Simo and Fox [4], Simo et al. [5], Bütcher et al. [6] or Braun et al. [7]. A general formulation for membranes based on curvilinear coordinates is given in Bonet et al. [8] and Lu et al. [9]. Taylor [10] proposed a large displacement finite element formulation of a membrane using three-noded triangular elements based on rectangular Cartesian coordinates. Details of the various terms involved are given in Valdés [11]. This work has been generalized for different finite elements by Rossi [12].

Some membrane structures have a very low flexural stiffness that can support a small amount of compressive stress before buckling appears. In order to avoid compression stresses, membranes are prestressed. Levy and Spillers [13], Raible [14] and Gil [15] use a prestressed method to analyze membranes which are initially flat in the Euclidean space. An approach that includes curved prestressed membranes using a projection scheme can be found in Bletzinger and Wüchner [16].

In the present work, the analysis of initially curved prestressed membranes is performed using a fiber orientation strategy, which is an extension of the work of Valdés et al. [17,18]. A deep study for prestressed and orthotropic membranes can be found in Valdés [19]. The fiber orientation approach here presented allows to analyze orthotropic membranes.

2. Formulation

For the membrane formulation a *curvilinear coordinate system* based on differential geometry of surfaces is used [9,16,20]. Greek indices on the membrane mid-surface take on values of 1 and 2 in a plane stress state in the Euclidean space.

The position vector $\mathbf{X}$ on the surface in the reference configuration Ω_0 is defined by two independent curvilinear coordinates ξ^1 and ξ^2, shown in Fig. 1, as

$$\mathbf{X} = \mathbf{X}(\xi^1, \xi^2) \tag{1}$$

The position vector $\mathbf{x}$ on the surface in the current configuration Ω is given by

$$\mathbf{x} = \mathbf{x}(\xi^1, \xi^2, t) \tag{2}$$

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doi:10.1016/j.finel.2008.11.008

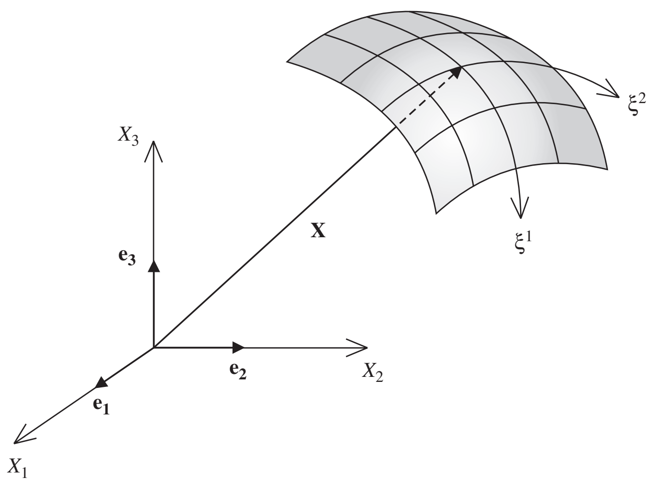

Fig. 1. Curvilinear coordinates for a surface.

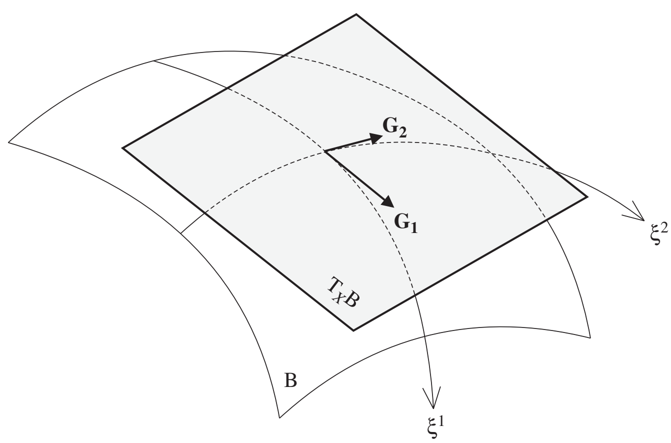

Fig. 2. Covariant base vectors forming a tangent plane.

The convected *covariant base vectors* of the curvilinear coordinate system on Ω_0 and Ω are defined, respectively, as

$$\mathbf{G}_\alpha = \frac{\partial \mathbf{X}}{\partial \xi^\alpha}, \quad \mathbf{g}_\alpha = \frac{\partial \mathbf{x}}{\partial \xi^\alpha} \tag{3}$$

where the covariant base vectors $\mathbf{G}_\alpha$ and $\mathbf{g}_\alpha$ form a tangent space $\mathrm{T}_X\mathscr{B}$ to the membrane surface and in general they are neither unit vector nor orthogonal to each other, as shown in Fig. 2.

The surface normals are defined by

$$\mathbf{G}_3 = \mathbf{G}_1 \times \mathbf{G}_2, \quad \mathbf{N} = \frac{\mathbf{G}_3}{\|\mathbf{G}_3\|}, \quad \mathbf{g}_3 = \mathbf{g}_1 \times \mathbf{g}_2, \quad \mathbf{n} = \frac{\mathbf{g}_3}{\|\mathbf{g}_3\|} \tag{4}$$

in the reference and current configurations, respectively. The normals are normalized given a unit vector. The covariant components of the metric tensors are defined by

$$G_{\alpha\beta} = \mathbf{G}_\alpha \cdot \mathbf{G}_\beta, \quad g_{\alpha\beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta \tag{5}$$

for the reference and current configurations, respectively. The convected *contravariant base vectors* at Ω_0 and Ω are given, respectively, by

$$\mathbf{G}^\alpha = G^{\alpha\beta} \cdot \mathbf{G}_\beta, \quad \mathbf{g}^\alpha = g^{\alpha\beta} \cdot \mathbf{g}_\beta \tag{6}$$

where the contravariant components of the metric tensors are obtained from

$$[G^{\alpha\beta}] = [G_{\alpha\beta}]^{-1}, \quad [g^{\alpha\beta}] = [g_{\alpha\beta}]^{-1} \tag{7}$$

for the corresponding configurations. When the contravariant base vectors are known, the covariant base vectors can be obtained from

$$\mathbf{G}_\alpha = G_{\alpha\beta} \cdot \mathbf{G}^\beta, \quad \mathbf{g}_\alpha = g_{\alpha\beta} \cdot \mathbf{g}^\beta \tag{8}$$

for the reference and current configurations, respectively. The covariant and contravariant base vectors define the scalar product identities

$$\mathbf{G}^\alpha \cdot \mathbf{G}_\beta = \delta^\alpha_\beta, \quad \mathbf{g}^\alpha \cdot \mathbf{g}_\beta = \delta^\alpha_\beta \tag{9}$$

where the Kronecker delta is given by

$$\delta^\alpha_\beta = \begin{cases} 1 & \text{when } \alpha = \beta \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

The deformation gradient tensor $\mathbf{F}$ in curvilinear coordinates is

$$\mathbf{F} = \mathbf{g}_\alpha \otimes \mathbf{G}^\alpha, \quad \mathbf{F}^T = \mathbf{G}^\alpha \otimes \mathbf{g}_\alpha, \quad \mathbf{F}^{-1} = \mathbf{G}_\alpha \otimes \mathbf{g}^\alpha, \quad \mathbf{F}^{-\mathrm{T}} = \mathbf{g}^\alpha \otimes \mathbf{G}_\alpha \tag{11}$$

Substituting Eq. (11) into the Green–Lagrange strain tensor yields

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) = \tfrac{1}{2}(\mathbf{G}^\alpha \otimes \mathbf{g}_\alpha \cdot \mathbf{g}_\beta \otimes \mathbf{G}^\beta - G_{\alpha\beta}\mathbf{G}^\alpha \otimes \mathbf{G}^\beta) \tag{12}$$

which components for the membrane surface in a plane stress state are

$$\mathbf{E} = E_{\alpha\beta}\mathbf{G}^\alpha \otimes \mathbf{G}^\beta, \quad E_{\alpha\beta} = \tfrac{1}{2}(g_{\alpha\beta} - G_{\alpha\beta}) \tag{13}$$

Using an appropriate constitutive equation to relate the second Piola–Kirchhoff stress tensor and the Green–Lagrange strain tensor in curvilinear coordinates, the components of the stress tensor are defined as

$$\mathbf{S} = S^{\alpha\beta}\mathbf{G}_\alpha \otimes \mathbf{G}_\beta \tag{14}$$

Finally, the virtual internal work is

$$\delta\mathscr{W}^{\mathrm{int}} = \int_{\Omega_0} \delta E_{\alpha\beta} S^{\alpha\beta}\,\mathrm{d}\Omega_0 \tag{15}$$

where all the tensor components are expressed in curvilinear coordinates.

2.1. Pressure follower forces

An important case for geometrically nonlinear analysis of membrane structures is that of uniform normal pressure follower forces. These forces change their direction each time the normal to the surface changes in the current configuration.

Consider a membrane element with an applied uniform pressure p acting on the current configuration having a pointwise normal $\mathbf{n}$. Then the traction force vector $\mathbf{t}$ is expressed as $p\mathbf{n}$, and the corresponding virtual external work in the current configuration is

$$\delta\mathscr{W}^{\mathrm{ext}} = \int_\Gamma \delta\mathbf{u} \cdot p\mathbf{n}\,\mathrm{d}\Gamma \tag{16}$$

3. Fiber orientation

The idea for the *fiber orientation* approach comes from the manufacturing process of membrane structures built with orthotropic or composite materials. A reference principal fiber direction is needed in these cases to perform the finite element analysis correctly. Even for isotropic materials the reference principal fiber direction is needed if the membrane has an initial *prestressed* field. With the methodology proposed here, a prestressed field for orthotropic materials is also possible.