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Convergence properties of materially and geometrically non-linear finite-element spatial beam analysis

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ABSTRACT

The way the non-linear constitutive equations in the spatial beam formulations are solved, influences the rate of convergence and the computational cost. Three different approaches are studied: (i) the direct global approach, where the constitutive equations are taken to be the iterative part of the global governing equations, (ii) the local (or indirect global) approach, where the constitutive equations are solved separately in each step of the global iteration, and (iii) the partly reduced approach, which is the combination of (i) and (ii). The approaches are compared with regard to the number of global iterations and the total number of floating point operations. The direct global approach is found to be the best choice.

1. Introduction

In the analysis of structures the non-linearity of the material law often dictates the rate of convergence of the solution procedure. The objective of the present paper is to find out how various approaches for solving the non-linear constitutive equations effect the rate of convergence and the total number of floating point operations in the finite-element analysis of spatial beams and frames. Several approaches for the solution of the discretized equations of the non-linear spatial beams are at hand. The constitutive equations, for example, can be eliminated from the set of the governing equations prior to its solution. This reduces the size of the global system of linearized equations that need to be solved in each iteration. When material is non-linear, such a formulation requires solving, in each step of the global iteration and locally in each integration point, the non-linear constitutive equations. This implies that, in each step of the global iteration, an additional number of local iterations need to be executed. This may remarkably increase the computational time. If, in contrast, the constitutive equations in integration points are assumed to be the part of the global governing equations, the size of the overall system of equations is larger. Such an approach may somewhat increase the number of global iterations, but because no additional local iterations are needed, the overall computational cost will probably be lower. Another interesting approach is a mixed approach, where some constitutive equations are solved locally, while the remaining equations become a part of global governing equations.

The influence of three different approaches on the rate of convergence and the overall computational cost is examined through numerical examples. We compare not only the number of iterations, but also the total number of floating point operations, thus assessing also the actual computational cost of the algorithm apart from the actual computer used.

2. Governing equations of the beam

The complete set of the beam equations consists of the cross-sectional constitutive equations (1) and (2), the equilibrium equations (3) and (4), the kinematic equations (5) and (6) (see [1–3])

$$\mathbf{R}(x)\mathscr{C}_N(\gamma_G(x),\boldsymbol{\kappa}_G(x)) - \boldsymbol{N}_g(x) = \mathbf{0} \tag{1}$$

$$\mathbf{R}(x)\mathscr{C}_M(\gamma_G(x),\boldsymbol{\kappa}_G(x)) - \boldsymbol{M}_g(x) = \mathbf{0} \tag{2}$$

$$\boldsymbol{N}'_g(x) + \boldsymbol{n}_g(x) = \mathbf{0} \tag{3}$$

$$\boldsymbol{M}'_g(x) + \boldsymbol{m}_g(x) - \boldsymbol{N}_g(x) \times \mathbf{R}(x)(\gamma_G(x) - \boldsymbol{c}_G(x)) = \mathbf{0} \tag{4}$$

$$\boldsymbol{r}'_g(x) - \mathbf{R}(x)(\gamma_G(x) - \boldsymbol{c}_G(x)) = \mathbf{0} \tag{5}$$

$$\boldsymbol{\vartheta}'_g(x) - \mathbf{T}^{-\mathrm{T}}(x)(\boldsymbol{\kappa}_G(x) - \boldsymbol{d}_G(x)) = \mathbf{0} \tag{6}$$

and the related static boundary conditions:

$$\boldsymbol{S}^0 + \boldsymbol{N}_g(0) = \mathbf{0}$$

$$\boldsymbol{P}^0 + \boldsymbol{M}_g(0) = \mathbf{0}$$

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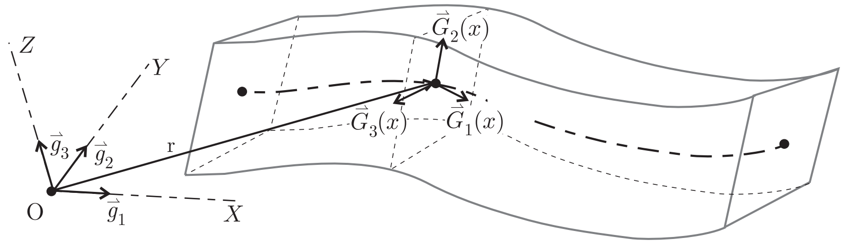

Fig. 1. Model of the 3D beam.

$$S^L - N_g(L) = \mathbf{0}$$

$$P^L - M_g(L) = \mathbf{0}$$

Here, the prime (′) denotes the derivative with respect to the arc-length parameter of the line of centroids in the initial configuration, x, and "×" marks the cross-vector product. The meaning of the notations used in the above equations is described below (see also Fig.1):

- g fixed (inertial) orthonormal basis $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ spanning the physical space of the beam;
- G orthonormal basis $\{\vec{G}_1, \vec{G}_2, \vec{G}_3\}$ spanning the cross-sectional planes;
- $\boldsymbol{N}, \boldsymbol{M}$ stress-resultant force and moment vectors over the cross-section;
- $\mathscr{C}_N, \mathscr{C}_M$ operators describing material of the beam;
- $\boldsymbol{\gamma}$ translational strain vector (γ_{G1} is the extensional strain, γ_{G2}, γ_{G3} are shear strains);
- $\boldsymbol{\kappa}$ rotational strain vector (κ_{G1} is the torsional strain, κ_{G2}, κ_{G3} are the curvatures);
- $\boldsymbol{r}$ position vector of the line of centroids of the beam;
- $\mathbf{R}$ both the rotation matrix from g to G and the coordinate transformation matrix ($\boldsymbol{v}_g = \mathbf{R}\boldsymbol{v}_G$);
- $\boldsymbol{\vartheta}$ rotational vector whose axis coincides with the axis of rotation and whose length equals the angle of rotation;
- $\boldsymbol{\Theta}$ skew-symmetric matrix $\boldsymbol{\Theta}$ composed from its axial vector $\boldsymbol{\vartheta}_g = [\vartheta_{g1}\ \vartheta_{g2}\ \vartheta_{g3}]^{\mathrm{T}}$;
- $\mathbf{T}^{\mathrm{T}}$ transformation matrix between $\boldsymbol{\kappa}_G$ and $\boldsymbol{\vartheta}'_g$ (i.e., $\mathbf{T}^{\mathrm{T}} = \mathbf{I} - ((1-\cos\vartheta)/\vartheta^2)\boldsymbol{\Theta} + ((\vartheta - \sin\vartheta)/\vartheta^3)\boldsymbol{\Theta}^2$, $\vartheta = \|\boldsymbol{\vartheta}_g\|$);
- $\boldsymbol{c}, \boldsymbol{d}$ variational constants determined from the known strains, position vectors and rotations in the initial configuration;
- $\boldsymbol{n}, \boldsymbol{m}$ external distributed force and moment vectors per unit of the undeformed length of the axis;
- $\boldsymbol{S}^0, \boldsymbol{S}^L$ external point forces at the boundaries $x = 0$, $x = L$;
- $\boldsymbol{P}^0, \boldsymbol{P}^L$ external point moments at the boundaries $x = 0$, $x = L$.

3. Reinforced concrete 3D beams

Due to its widespread use in practice the reinforced concrete material is found convenient for demonstrating the convergence properties of various approaches. Let us describe the mathematical model of reinforced concrete first.

3.1. Constitutive law of concrete

We follow Desayi and Krishnan [4] and Bergan and Holand [5] and employ the uniaxial stress–strain relation for concrete given by the function (see Fig. 2 for its graph):

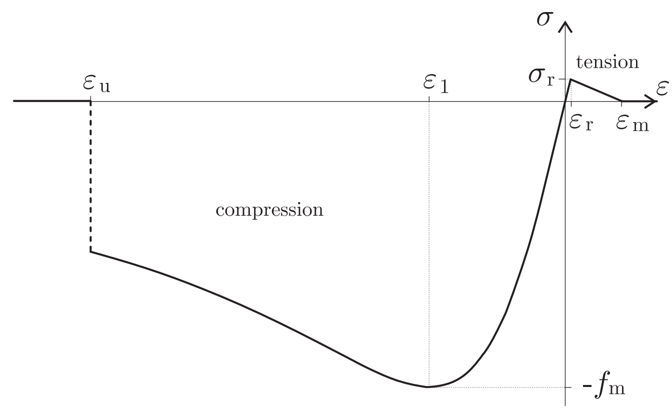

Fig. 2. Constitutive law of concrete.

$$\sigma(\varepsilon) = \begin{cases} 0, & \varepsilon \leqslant \varepsilon_u \\ 2f_m|\varepsilon_1|\dfrac{\varepsilon}{\varepsilon_1^2 + \varepsilon^2}, & \varepsilon_u < \varepsilon \leqslant \varepsilon_r \\ \dfrac{\sigma_r}{\varepsilon_r - \varepsilon_m}(\varepsilon - \varepsilon_m), & \varepsilon_r < \varepsilon \leqslant \varepsilon_m \\ 0, & \varepsilon_m < \varepsilon \end{cases} \tag{7}$$

Here f_m is ultimate strength of concrete in compression ($f_m > 0$); $\varepsilon_1 < 0$ is the corresponding strain; $\varepsilon_u < 0$ is ultimate strain in compression; $\varepsilon_r > 0$ is strain at ultimate strength of concrete in tension and $\varepsilon_m > 0$ is ultimate strain in tension. Parameters f_m, ε_1, and ε_u have to be determined in compression tests on concrete cylinders; similarly ε_r and ε_m have to be determined by tension tests. Because the tension zone is not essential in modelling concrete, the tension tests are rarely performed in practice and empirically found average values $\varepsilon_r = 5.5 \times 10^{-5}$ and $\varepsilon_m = 7 \times 10^{-4}$ are utilized instead [5].

Note that the function proposed in Eq. (7) is discontinuous at ε_u and that its first derivative with respect to ε is discontinuous at ε_u, ε_r and ε_m. This property is very inconvenient, because the formation of the tangent stiffness matrix of the beam element is then faced with the integration of a discontinuous function over the cross-section, which requires special measures to be applied.

In spatial beam elements, when used in frame-like structures, we usually assume the Bernoulli hypothesis that a cross-section suffers only rigid translation and rotation during deformation. This implies that the normal strain (axial strain), ε, at the cross-section (x=*const.*) is linearly distributed over the cross-section:

$$\varepsilon(y, z) = \gamma_{G1} - y\kappa_{G3} + z\kappa_{G2} \tag{8}$$

Here, y and z are the local coordinates at the cross-section defined by base vectors $\vec{G}_2$ and $\vec{G}_3$ (see also Fig. 1). The corresponding normal (axial) stress distribution, $\sigma(y, z)$, over the concrete cross-section $x =$ *const.* is determined from the constitutive law (7). The integration of