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Incompressible flow strategies for ice creep

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ABSTRACT

The objectives of this paper are to examine the use of volumetric strain rate and pressure enhancement strategies for low order finite elements, and to present a stable matrix-free algorithm for solving steady-state flow problems. Algorithms based on the method of successive approximation and low order finite elements are examined for determining the steady-state flow field of a boundary-valued problem consisting of an incompressible material. It is shown that both volumetric strain rate and pressure enhancement are required to mitigate pathological locking and nonphysical pressure variations. Care must however be taken when introducing pressure enhancement, which helps mitigate the pressure from drifting, as the stress field is perturbed from equilibrium. An algorithm based on dynamic relaxation and radial return stress calculations is presented for matrix free calculations dealing with stress-dependent creep.

1. Introduction

There are many problems in geotechnics, including glacier flow, landslides and debris flows, that involve slope creep, cf. [1]. Analysis of ice mechanics problems is of interest to mining companies because considerable economic resources in terms of mineral deposits are at the margins or under ice sheets. Estimates of ice flow towards a pit are required to assess operational efforts and mine economics [2]. Colbeck [3] in 1973, for example, carried out a flow analysis of ice into a proposed open-pit, iron ore mine at the edge of the Greenland ice sheet. The objective of the study was to establish the most favorable profile for the ice sheet next to the mine to minimize the annual amount of ice that would have to be removed. Owing to global warming and its effect on climate change and the retreat of ice, interest in resource and economic development near or at the fringes of ice sheets has increased.

The flow of ice is often treated as a viscous, nonlinear, incompressible fluid that obeys Stoke's flow equation [4–6]. Such problems can be challenging from a numerical point of view, particularly when using low order finite elements, which are attractive when dealing with a large number of unknowns together with iterative solvers. The objectives of this paper are to examine the use of volumetric strain rate ($\dot{\varepsilon}_v$) and pressure (p) enhancement strategies for low order finite elements, and to present a stable, matrix-free algorithm for solving steady-state flow problems based on combining the method of successive approximation with dynamic relaxation.

We begin by defining the class of the problem that is of interest and briefly discuss the solution of Stoke's flow equation. Thereafter the proposed procedure is presented, including enhancement techniques that are required to suppress pathological locking and nonphysical pressure variations that are common when using low-order elements. An example is presented to highlight some of the issues and to show that care must be taken when implementing enhancement techniques. The literature in this area is vast, thus only that most relevant is cited.

2. Problem definition and field equations

We are interested in solving for the steady-state, two-dimensional flow conditions in a large ice mass as represented by the schematic of Fig. 1. To determine the stress and velocity fields, it is necessary to consider the relation between strain rate and velocity, the relation between stress and strain rate, the momentum balance and mass balance, as well the boundary conditions. It will be assumed that the ice is incompressible and that the 'elastic' strains are negligible for steady-state creep when compared to the creep strains, which is why ice can be treated as a very viscous fluid.

Kinematics: The velocity $\boldsymbol{v} = \dot{\boldsymbol{u}}$ of the ice at a location is defined by position vector $\boldsymbol{x} = x_i \vec{e}_i$ where x_i are the coordinates and $\vec{e}_i$ represent the orthogonal basis vectors in three-dimensional space. Repeated indices imply summation. The superposed dot indicates

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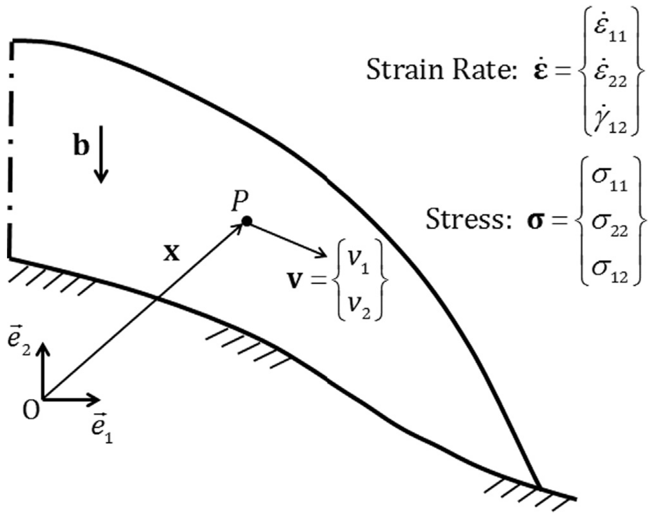

Fig. 1. Definition of problem.

a derivative with respect to time t, in this case the derivative of displacement $\boldsymbol{u}$.

Stress: Adopting Voigt notation for second order tensors with bold symbols denoting vectors or matrices, stress $\boldsymbol{\sigma} = \langle \sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{23}\ \sigma_{31} \rangle^T$ can be decomposed into its pressure p and deviatoric $\mathbf{S}$ components according to

$$\boldsymbol{\sigma} = -p\mathbf{m} + \mathbf{S} \tag{1}$$

in which $\mathbf{m} = \langle 1\ 1\ 1\ 0\ 0\ 0 \rangle^T$. Tensile stresses are taken as positive, with positive pressure implying compression.

Strain and strain rate: Given that we are interested in steady-state solutions, no distinction is made here between strain rate $\dot{\boldsymbol{\varepsilon}}$ and rate of deformation, although strictly speaking the two are not the same; see, e.g., [7]. Engineering strain $\boldsymbol{\varepsilon} = \langle \varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \gamma_{12}\ \gamma_{23}\ \gamma_{31} \rangle^T$ is determined from the displacement gradients via $\boldsymbol{\varepsilon} = \boldsymbol{Lu}$ with the strain rate given by $\dot{\boldsymbol{\varepsilon}} = \boldsymbol{Lv}$ in which the linear operator $\mathbf{L}$ is defined according to

$$\boldsymbol{L}^T = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix} \tag{2}$$

with the superscript T implying transposition. For two-dimensional flow problems, the out-of-plane strain rate components are zero and are therefore not required.

With regard to discrete motion, the displacement at time t^n is given by $\boldsymbol{u}^n$ with the change in displacement defined as $\Delta\boldsymbol{u} = \Delta t \boldsymbol{v}^{n+1}$ such that $\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \Delta\boldsymbol{u}$ at t^{n+1} in which the superscript n is a step counter and symbol Δ denotes increment.

Constitutive relation: Restricting ourselves to incompressible, two-dimensional flow with pressure insensitive, isotropic material behaviour, the relation between deviatoric stress and strain rate takes the form

$$\mathbf{S} = \mathbf{D}_d[\dot{\boldsymbol{\varepsilon}} - \tfrac{1}{3}\mathbf{m}\mathbf{m}^T\dot{\boldsymbol{\varepsilon}}] \tag{3}$$

with

$$\mathbf{D}_d = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

in which μ is the viscosity and $\mathbf{S} = \langle S_{11}\ S_{22}\ S_{12} \rangle^T$. The out-of-plane deviatoric stress is given by $S_{33} = -(S_{11} + S_{22}) \to 0$ for two-dimensional flow. For incompressible problems the volumetric strain rate $\mathbf{m}^T\dot{\boldsymbol{\varepsilon}}$ vanishes; thus the deviatoric strain rate is the same as the total strain rate. For nonlinear flow it is convenient to use Glen's [8] power law $\dot{\varepsilon}_e = A\sigma_e^r$ to define the viscosity as

$$\mu = \frac{1}{3}\frac{\sigma_e}{\dot{\varepsilon}_e} \to \mu = \frac{1}{3A}\sigma_e^{1-r} \tag{5}$$

where $\sigma_e = \sqrt{\frac{3}{2}\mathbf{S}^T\mathbf{S}}$ and $\dot{\varepsilon}_e = \sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}_d^T\dot{\boldsymbol{\varepsilon}}_d}$ are Dorn's definitions for the equivalent stress and equivalent strain rate, respectively, with the subscript d implying deviator strain, and r and A being material properties that depend on temperature, cf. [9,10].

Conservation equations: Keeping in mind that we are treating the solid as a very viscous fluid, let us begin by writing the momentum balance:

$$\rho\dot{\mathbf{v}} = \boldsymbol{L}^T\boldsymbol{\sigma} + \mathbf{b} \to 0 \tag{6}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, ρ is the mass density, and $\boldsymbol{g}$ is the gravitational acceleration vector. All variables are a function of position $\boldsymbol{x}$ and time t. Eq. (6) is assumed to correspond to a spatial description. Although the acceleration term has been maintained, it vanishes for very slow creep rates. Since we are dealing with creeping flow, there is no need to include the advection term for acceleration.

The second balance equation is a statement of mass balance:

$$\dot{p} = -K\mathbf{m}^T\dot{\boldsymbol{\varepsilon}} \to 0 \tag{7}$$

in which K is a 'bulk modulus' that is often expressed in terms of the material's wave speed c_p, i.e., $K = \rho c_p^2$. Changes in pressure are naturally related to changes in volumetric strain. For quasi-static equilibrium conditions the pressure rate also vanishes. Both $\dot{p}$ and $\dot{\mathbf{v}}$ have been retained in the relations as they are required in the transient relaxation schemes. To develop a unique solution for a particular boundary-valued problem, we require boundary conditions. For a free surface the tractions are zero, $\mathbf{n}^T\boldsymbol{\sigma} = \mathbf{0}$, as are the velocities, $\mathbf{v} = \mathbf{0}$, if the boundary is fully fixed. The matrix $\mathbf{n}$ contains the normal to a surface. At a divide we have mixed conditions where both shear traction and normal velocity are zero along the boundary.

Owing to the similarity between the equations for a very viscous fluid and those of a creeping solid, both balance equations also apply for creeping solids that undergo small changes in geometry; in the latter case it is understood that the derivatives are with respect to the initial configuration, and Eq. (7) appears as part of the constitutive equation.

3. Solution schemes

The principle of virtual velocities [7] together with a constraint equation may be used to convert Eq. (6) and (7) to integral forms that are suitable to develop the finite element matrices, see, e.g., [11]. Given volume V, we have

$$\int_V \delta\mathbf{v}^T\rho\dot{\mathbf{v}}\, dV = \int_V \delta\dot{\boldsymbol{\varepsilon}}^T\boldsymbol{\sigma}\, dV + \int_V \delta\mathbf{v}^T\boldsymbol{b}\, dV + \int_{S_t} \delta\mathbf{v}^T\boldsymbol{t}\, dS \tag{8}$$

and

$$\int_V \delta p\dot{p}\, dV = -\int_V \delta p\mathbf{m}^T\dot{\boldsymbol{\varepsilon}}\, dV \tag{9}$$

where $\boldsymbol{t}$ is the prescribed traction on boundary S_t, and the symbol δ represents a virtual quantity. These equations can be shown to correspond to a variational principle for incompressible creeping flow [12]. It is important to recognize that the specification of pressure on a boundary enters through the traction term and cannot be specified as an essential (fixed) condition.