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A gradient smoothing method (GSM) based on strong form governing equation for adaptive analysis of solid mechanics problems

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ABSTRACT

A gradient smoothing method (GSM) based on strong form of governing equations for solid mechanics problems is proposed in this paper, in which gradient smoothing technique is used successively over the relevant gradient smoothing domains to develop the first- and second-order derivative approximations by calculating weights for a set of field nodes surrounding a node of interest. The GSM is found very stable and can be easily applied to arbitrarily irregular triangular meshes for complex geometry. Unlike other strong form methods, the present method has excellent stability that is crucial for adaptive analysis. An effective and robust residual based error indicator and simple refinement procedure using Delaunay diagram are then implemented in our GSM for adaptive analyses. The reliability and performance of the proposed GSM for adaptive procedure are demonstrated in several solid mechanics problems including problems with singularities and concentrated loading, compared with the well-known finite element method (FEM).

1. Introduction

With the rapid development of computer technology in the past few decades, a broad range of numerical methods have been developed for different types of problems and achieved great success, e.g., the finite element method (FEM), finite difference method (FDM), finite volume method (FVM) and recently the meshfree methods [1–3]. In advanced design of products of high precision, adaptive analysis is becoming an important tool in practical numerical computations [4]. It is a fundamental tool to obtain numerical solutions with a desired accuracy. In an adaptive procedure, there are three essential ingredients: (1) an effective and stable numerical method for arbitrary problem domains and irregular meshes; (2) a tool for estimating the error of the numerical solution; and (3) an algorithm to refine the problem domain. The first ingredient is a prerequisite, without which an adaptive process will break down. The error estimator is crucial in assessing the local and global errors in the numerical solution at a stage of analysis, whereby a decision can be made on whether a refinement is required. The third is performed according to the error information provided by the error estimate. The effectiveness and efficiency of all these three pieces of techniques are critical to the performance of an adaptive analysis. To conduct a posteriori error estimation, two values of a quantity—a computed value and a reference value—are usually required. The first is the raw data of the numerical solution while the second is derived from the raw data via postprocessing (smoothing or projection). In FEM, it is well known that the raw stresses (or derivatives) do not possess inter-element continuity and have a low accuracy at nodes and element boundaries. The improved values are obtained by smoothing the inter-element discontinuity. The difference between the raw and improved values forms a basis for error estimation in FEM solution. Detailed descriptions of this approach can be found in FEM literatures, e.g., by Zienkiewicz [5].

To establish an adaptive finite element procedure, one of the most important components is a robust automatic mesh generation scheme. However, to develop and implement automatic mesh generators with good control of element size and shape is not an easy task. During the last decade, many research efforts have been devoted to this area [6,7] and yet it still remains an active research topic in computational mechanics and geometry. Currently, automatic mesh generators of triangular elements for complex geometry are available. Unfortunately, the triangular elements used in FEM are known to be 'too stiff' and inaccurate. Compared with the finite element method, the meshfree methods enjoy much more flexibility

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in model generation since they can approximate field variables entirely based on a group of discrete nodes and require no predefined node connectivity. For meshfree methods that require background cells, triangular cells can be used, which will not affect the accuracy in the solutions. Nodes used in many meshfree methods can be irregular or unstructured. Nodes can be quite freely inserted or deleted without worrying too much about the connectivities. Therefore, the meshfree methods are particularly attractive for the development of adaptive strategies. Several adaptive procedures and error estimates for meshfree methods have been proposed. Duarte and Oden [8] derived an error estimator for the *h-p* cloud methods that involves only the computation of interior residuals and residuals where Neumann boundary conditions are prescribed. Liszka et al. [9] built discrete models of boundary-value problems with different adaptive strategies. Chung et al. [10], Gavete et al. [11] and Lee and Zhou [12] proposed adaptive refinement procedures and error indicators for the element-free Galerkin (EFG) method. Park et al. [13] developed a posteriori error estimates and an adaptive refinement scheme of first-order least-squares meshfree method.

Among these developed adaptive meshfree methods, the weak form methods, e.g., EFG method, are most well established. The solutions of weak form methods are usually very stable. In contrast, the development of meshfree strong form methods is rather sluggish. Available literatures for the meshfree strong form methods are still very limited. However, the meshfree strong form method possesses many good features for adaptive analysis due to its simplicity. The strong formulation is much simple, straightforward and easy to implement. The meshfree strong form method is considered a truly meshfree method as it does not even require background cells that are needed in weak form method for integration. Such distinct features facilitate the refinement or coarsening scheme in the adaptive scheme. Moreover, unlike weak form methods, strong form methods need no integration and hence no mapping is needed.

However, the instability problem has been a key factor that limits the application of meshfree strong form methods that use local nodes. Researchers have introduced several stabilization schemes [14,15], in which stabilization factors need to be determined. Many efforts have been devoted to point collocation methods based on reproducing kernel approximations [16–18]. Currently, most of the 'full-proof' strong form method is still very much relying on the structured grid and restricted regular domain. Although methods like generalized finite difference method (GFDM) [19,20] can be used for irregular domain and unstructured grid, a proper stencil (node selection) is somehow still needed for function approximation. Such inconvenience procedures give difficulties to the strong form method in the adaptive process. In addition, since nodal distribution during the adaptation can become highly irregular, a 'proper' stencil can be costly and difficult to form.

In this paper, a gradient smoothing method (GSM) is proposed based on strong form governing equations. Gradient smoothing technique is utilized to construct first- and second-order derivative approximations by systematically computing weights for a set of nodal points surrounding a node of interest. Three types of different domains for the gradient smoothing operations are devised. The strong form of governing equations is directly discretized at nodes using gradient smoothing repeatedly over relevant gradient smoothing domains. These computations can be easily performed based on an irregular triangular mesh that can be generated automatically for complex geometries. The stencil analyses of weighting coefficients have been conducted for the Laplace operators, and favorable weight distributions are found. The proposed GSM can effectively overcome the instability issue, while retaining the strong form feature of simplicity in formulation procedures which is particularly suitable for adaptive analysis.

A residual based error indicator is then adopted in our GSM for adaptive analyses. By evaluating the residual of the governing equation for each triangular cell in the domain, error indicator effectively identifies the necessary regions to be refined. Simple refinement procedure using Delaunay diagram is adopted in the adaptive scheme. Additional nodes can be inserted into the domain easily without worrying about the nodal connectivity and remeshing the domain.

The layout of this paper is as follows: Section 2 theoretically formulates the GSM. In Section 3, a brief description of a posteriori error indicator based on residual of the governing equation is provided. Section 4 illustrates the capabilities of the present method through some numerical examples including different levels of stress concentration. The performance of the proposed strategy is also assessed by comparing the convergence rate obtained with those by uniform refinement. Conclusions are stated in Section 5.

2. Gradient smoothing method (GSM)

2.1. Gradient smoothing

Consider a two-dimensional elastostatic problem governed by the following equilibrium equation in the domain Ω:

$$\mathbf{Lu} = \mathbf{f} \quad \text{in } \Omega \tag{1}$$

with essential (Dirichlet) boundary conditions

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u \tag{2}$$

and natural (Neumann) boundary conditions

$$\mathbf{Bu} = \mathbf{g} \quad \text{on } \Gamma_t \tag{3}$$

where $\mathbf{L}$, $\mathbf{B}$ are the differential operators, $\mathbf{u}$ is the field variable and $\mathbf{f}$, $\mathbf{g}$ are external force vectors. Eq. (3) is derived using Cauchy's formula

$$\sigma_{ij} n_j - t_i = 0 \tag{4}$$

In the strong form methods, Eqs. (1)–(3) are directly collocated at the field nodes in the problem domain and on the boundaries, respectively. The discretized system governing equations are given as follows:

$$L(u_i) = f_i \quad \text{in } \Omega \tag{5}$$

with Dirichlet boundary conditions

$$u_i = \bar{u}_i \quad \text{on } \Gamma_u \tag{6}$$

and Neumann boundary conditions

$$B(u_i) = g_i \quad \text{on } \Gamma_t \tag{7}$$

where subscript "i" denotes the collocation point.

The governing equations (5)–(7) can be collocated at their corresponding field nodes then be assembled and expressed in the following matrix form:

$$\mathbf{KU} = \mathbf{F} \tag{8}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{F}$ is the force vector and $\mathbf{U}$ is the vector of unknown nodal values. Note that the stiffness matrix resulted from collocation is generally unsymmetric. The vector of unknown nodal values can be easily solved as

$$\mathbf{U} = \mathbf{K}^{-1}\mathbf{F} \tag{9}$$

if $\mathbf{K}$ is not singular and well-conditioned.

In the present method, the problem domain Ω is discretized by triangular cells as shown in Fig. 1. For the i-th node, a smoothing domain Ω_i is generated by sequentially connecting the centroids with mid-edge points of surrounding triangular cells. Γ_i is the boundary