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A plane strain quadrilateral Cosserat point element (*CPE*) for nonlinear orthotropic elastic materials — An extension to initially distorted geometry and general orthotropic directions

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ABSTRACT

The objective of the present paper is to develop a four-noded quadrilateral Cosserat point element (*CPE*) for the numerical solution of plane strain problems in finite elasticity of orthotropic materials with general orientation and initially distorted geometry. Generally speaking, the Cosserat point approach connects the kinetic quantities to derivatives of a strain energy function and once the strain energy of the *CPE* has been specified, the procedure needs no integration over the element region and it ensures that the response of the *CPE* is hyperelastic. In the present paper a functional form for the strain energy that controls the inhomogeneous deformations and allowing for accurate modeling of distorted meshes is proposed. A number of example problems, which compare the performance of the developed quadrilateral *CPE* with that of other elements based on the mixed formulation, reduced integration with enhanced hourglass control, and enhanced strains/incompatible modes methods, are considered. These examples demonstrate that *CPE* is free of hourglass instabilities, and it is a robust user friendly element that can be used for modeling finite deformations of orthotropic elastic materials.

1. Introduction

Renewed interest in modeling structures made out of anisotropic materials was motivated by the need for modeling biological tissues, which are reinforced by bundles, and modeling fiber reinforced rubber that can be used in the tire industry. However, the finite element method can be used for modeling both isotropic and anisotropic structures. It is well known that the Bubnov–Galerkin approach exhibits locking for bending dominated response of thin structures with poor element aspect ratios, and for nearly incompressible materials. Therefore, various element technologies including methods of mixed variational principles [27], hybrid formulation [16], reduced integration with hourglass control [3,4,17,18], enhanced assumed strains [24–26,29,5,14], and mixed-enhanced strains [11,12] were developed to overcome these deficiencies. Although each of the latter approaches overcomes one or more of the above-mentioned unphysical behaviors, they may have a number of drawbacks. For instance, one of the drawbacks of the reduced integration method with hourglass control is the difficulty in determining stabilization parameters in the hourglass scheme, and their possible unphysical influence on the solution. The elements based on the enhanced strain technique can exhibit unphysical hourglassing in regions experiencing combined high compression with bending. Furthermore, it was shown in Jabareen and Rubin [7] that some of these improved element formulations in commercial codes can exhibit inelastic response, even though they attempt to model a hyperelastic material with a strain energy function.

Bearing in mind the deficiencies of the above-mentioned finite element technologies, the emerging need for modeling structures made out of anisotropic materials sets a challenge to the scientific and engineering communities. Recently, a novel finite element technology based on the theory of a Cosserat point [19–21] was developed and applied in the formulation of a 3-D brick element for the numerical solution of dynamic problems for nonlinear hyperelastic materials [15]. By way of background, the kinematics of the brick (*CPE*) are characterized by eight element director vectors and the kinetics propose eight balance laws of director momentum to determine the dynamic response of the element. A novel feature of the *CPE* is that the element is considered to be a structure with a response characterized by its strain energy. The nodal forces are related to derivatives of the strain energy function through algebraic relations in a similar manner to the relationship of the stress to derivatives of the strain energy function in the full three-dimensional theory of hyperelastic materials.

The objective of the present paper is to generalize the *CPE* formulation presented in Jabareen et al. [10] by making it capable for analyzing general elastic orthotropic materials under finite

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deformations with initially distorted meshes. In particular, in Jabareen et al. [10] the coefficients of the strain energy function, which controls the response to inhomogeneous deformations, were determined by limiting attention to a rectangular parallelepiped, and it was found that the formulation therein suffers from undesirable sensitivity to initially distorted meshes. Furthermore, it was shown by Jabareen and Rubin [8,9] that generalizing the *CPE* to accurately model general element shapes is an extreme challenge, and therefore, the development presented in the present paper is focused on the two-dimensional plane strain case. However, the resulting two-dimensional *CPE* should have a wider range of applications, such as within realistic two-dimensional engineering problems.

An outline of the present paper is as follows: Section 2 presents the theoretical background of the two-dimensional *CPE*. Section 3 describes the procedure for determining the constitutive coefficients in the strain energy function for the inhomogeneous deformations, and Section 4 discusses the formulation of the numerical solution of plane strain problems using the *CPE*. Section 5 presents a number of plain strain examples that demonstrate the response of the developed quadrilateral *CPE* and comparing it to different finite elements using the commercial finite package ABAQUS. Finally, Section 6 presents the conclusions.

2. Theoretical background of the Cosserat point element (*CPE*)

The *CPE* for a plane strain quadrilateral element is a continuum model which characterizes the response of a finite structure that occupies the region *P*, which is a right cylinder with a quadrilateral cross-section and unit thickness (see Fig. 1). Unless otherwise stated, all vectors and tensors are two-dimensional with components in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane associated with the fixed right-handed orthonormal rectangular Cartesian base vectors $\mathbf{e}_i$ $(i=1,2,3)$.

2.1. Kinematics

The location $\mathbf{X}^*$ of a material point in the stress-free reference configuration of a quadrilateral *CPE* and the location $\mathbf{x}^*$ at time *t* of the same material point in the present configuration admit the representations

$$\mathbf{X}^* = \sum_{i=0}^{3} N^i(\theta^1,\theta^2)\mathbf{D}_i, \quad \mathbf{x}^* = \sum_{i=0}^{3} N^i(\theta^1,\theta^2)\mathbf{d}_i(t), \tag{1}$$

where $\{\mathbf{D}_i, \mathbf{d}_i\}$ $(i=0,1,2,3)$ are, respectively, the reference and present element director vectors, θ^i $(i=1,2)$ are convected coordinates having the ranges $-1/2 \le \theta^i \le 1/2$, and N^i $(i=0,1,2,3)$ are the bilinear shape functions defined by

$$N^0=1, \quad N^1=\theta^1, \quad N^2=\theta^2, \quad N^3=\theta^1\theta^2. \tag{2}$$

Furthermore, the reference element directors $\{\mathbf{D}_1,\mathbf{D}_2\}$ and the present element directors $\{\mathbf{d}_1,\mathbf{d}_2\}$ are restricted to be linearly independent so that

$$D^{1/2} = \mathbf{D}_1 \times \mathbf{D}_2 \cdot \mathbf{e}_3 > 0, \quad d^{1/2} = \mathbf{d}_1 \times \mathbf{d}_2 \cdot \mathbf{e}_3 > 0. \tag{3}$$

With reference to Fig. 1 it follows that the reference nodal directors $\overline{\mathbf{D}}_i$ $(i=0,\ldots,3)$ and their present counterparts $\overline{\mathbf{d}}_i$ $(i=0,\ldots,3)$, which locate the nodes of the quadrilateral in the reference and present configurations, respectively, are defined by the values of the nodal position vectors such that

$$\overline{\mathbf{D}}_i = \mathbf{X}^*(\theta^1=\theta^1_i, \theta^2=\theta^2_i), \quad \overline{\mathbf{d}}_i = \mathbf{x}^*(\theta^1=\theta^1_i, \theta^2=\theta^2_i, t), \tag{4}$$

where $\{\theta^1_i, \theta^2_i\}$ are the values of the convected coordinates at the *i*th node and are recorded in Table 1. The reference, $\mathbf{D}_i$, and the present, $\mathbf{d}_i$, element director vectors can be written in terms of the reference, $\overline{\mathbf{D}}_i$, and the present, $\overline{\mathbf{d}}_i$, nodal director vectors, respectively, by using the expressions in (4) such that

$$\mathbf{D}_i = \sum_{j=0}^{3} A_{ij}\overline{\mathbf{D}}_j, \quad \mathbf{d}_i = \sum_{j=0}^{3} A_{ij}\overline{\mathbf{d}}_j \quad (i=0,1,2,3), \tag{5}$$

where the constant matrix A_{ij} is defined by

$$[A_{ij}] = \frac{1}{4}\begin{bmatrix} +1 & +1 & +1 & +1 \\ -2 & +2 & +2 & -2 \\ -2 & -2 & +2 & +2 \\ +4 & -4 & +4 & -4 \end{bmatrix}. \tag{6}$$

The deformation of the quadrilateral *CPE* is characterized by a tensor $\mathbf{F}$ associated with homogeneous deformation, and one vector $\boldsymbol{\beta}$ associated with inhomogeneous deformation, such that

$$\mathbf{F} = \sum_{i=1}^{2} \mathbf{d}_i \otimes \mathbf{D}^i, \quad \boldsymbol{\beta} = \mathbf{F}^{-1}\mathbf{d}_3 - \mathbf{D}_3, \tag{7}$$

where $\mathbf{D}^i$ $(i=1,2)$ are the reciprocal vectors of $\mathbf{D}_i$ $(i=1,2)$ and $\otimes$ denotes the tensor product. Moreover, it is possible to show that the pointwise two-dimensional deformation gradient $\tilde{\mathbf{F}}^*$ can be

Table 1
Values of the convected coordinates at the nodes of the element.

Node	θ^i_1	θ^i_2
0	$-\frac{1}{2}$	$-\frac{1}{2}$
1	$+\frac{1}{2}$	$-\frac{1}{2}$
2	$+\frac{1}{2}$	$+\frac{1}{2}$
3	$-\frac{1}{2}$	$+\frac{1}{2}$

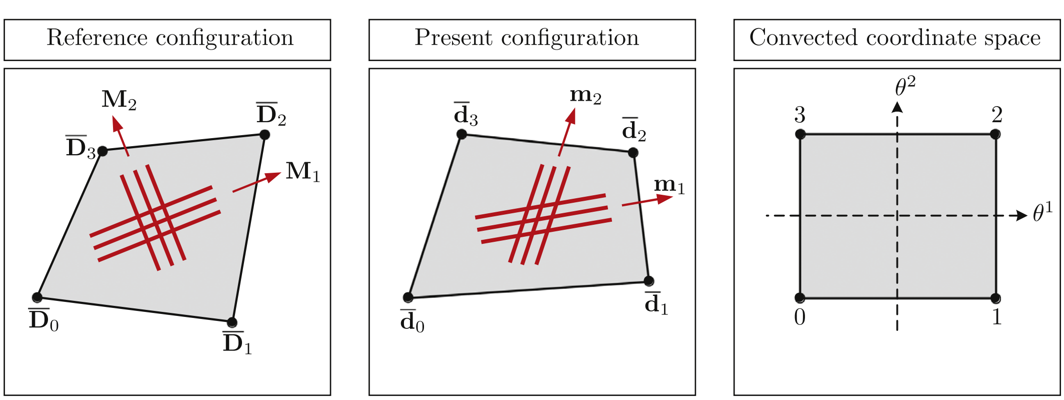

Fig. 1. Sketch of a general quadrilateral *CPE* showing the reference configuration, the present configuration, the convected coordinate space, the nodal numbering, and material directions.