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Weak form quadrature element analysis of spatial geometrically exact shear-rigid beams

Run Zhang, Hongzhi Zhong *

Department of Civil Engineering, Tsinghua University, Beijing 100084, PR China

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ABSTRACT

In this paper a total Lagrangian weak form quadrature element formulation of spatial shear-rigid beams undergoing large displacements and rotations is presented. A geometrically exact beam model with zero transverse shear deformation is adopted. Quaternion representation of finite rotations of spatial beams is used to avoid possible singularity in parameterization of rotation. The formulation reduces the number of degrees of freedom within the element as well as satisfies the demand of strain-objectivity. Several numerical examples are presented to illustrate the feasibility of the formulation.

1. Introduction

Geometrically nonlinear analysis of beams allowing for large displacements and rotations has been of concern in various engineering disciplines. Since Simo and Vu-Quoc [1,2] extended the pioneering work of Reissner [3] and developed the geometrically exact Reissner–Simo beam model, a great deal of research work in regard to static and dynamic analyses of beams undergoing large displacements and rotations has been done over the past few decades. In the Reissner–Simo beam model, extension, flexure, torsion and shearing are considered, being pertinent to the linear Timoshenko beam model. Corresponding to the linear Euler–Bernoulli beam model, a geometrically exact beam model excluding shearing can also be established. This shear-rigid beam model can be seen as a kinematical simplification of Reissner–Simo (shear-deformable) beam model. In spite of kinematical complexity, the shear-deformable beam model enjoys mathematical simplicity and elegance, as has been shown in the work of Reissner and Simo [1–3]. In contrast, the shear-rigid beam model is kinematically simple but it may run into mathematical complexity due to the zero shear deformation constraints. Nonetheless, it possesses significant advantages in nonlinear analysis of beams. First, it is well-known that the transverse shear deformation is trivial for slender beams despite large displacements and rotations. It is therefore possible to develop a numerical system with less number of degrees-of-freedom for the shear-rigid beam model. Second, precautions have to be taken to eliminate locking phenomenon for displacement-based formulations of the shear-deformable beam model while this is unnecessary for the shear-rigid beam model. Besides, the account of shear deformation may impose strong time stepsize restrictions in dynamic analysis of beams [4].

Although the majority of research work over the past three decades is about the shear-deformable beam model, the shear-rigid beam model still receives attention of researchers in nonlinear analysis of beams. By adopting modified Hu-Washizu variational principle, Saje proposed a finite element formulation for planar slender straight beams [5]. Gerstmayr and Shabana analyzed thin beams and cables using absolute nodal coordinates [6]. Zhao and Ren proposed a quaternion-based singularity-free slender beam element, using a sequential interpolating method to ignore transverse shear deformation [7]. A recent investigation of planar slender beams [8] indicates that nonlinear analysis of slender geometrically exact beams based on the shear-rigid beam model offers satisfactory results. The present paper is an endeavor to deal with spatial beams with large displacements and rotations based on the shear-rigid beam model.

Differing from planar beams considerably, spatial rotations of three-dimensional beams falls within multiplicative special orthogonal group and the configuration space of the beam is a nonlinear manifold. As a result, the accuracy, robustness and efficiency of numerical formulations are strongly dependent upon the parameterization and approximation of spatial rotations. Many representations for spatial rotations, including the nine-parameter orthogonal tensor, the three-parameter rotation vector, Rodrigues' parameters, quaternions, have been proposed. Detailed accounts of

* Corresponding author. Tel.: +86 10 62781891.
E-mail address: hzz@tsinghua.edu.cn (H. Zhong).

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these parameterizations may be found in [9,10]. For constrained rotations, Ibrahimbegović et al. used the incremental rotation vector to handle geometrically exact shell model with no drilling rotations [11]. Among these parameters, the qurternion-based rotation has been shown to be a convenient representation of spatial rotations. A recent study further highlights the advantages of the quaternion-based representation of spatial rotations [12]. In this paper, quaternions are used to express cross-section rotations and corresponding strain measures to avoid singularity problems that may be encountered when rotation vectors are used in a total-Lagrangian formulation.

Over the years, numerical methods adopting displacement-based formulations have been dominant in nonlinear analysis of geometrically exact beams despite the development of other formulations such as hybrid-mixed finite element formulation [13] and stress-based formulation [14]. In most finite element formulations wherein low order displacement-based finite elements are used for shear-deformable beams, appropriate procedures have to be taken to eliminate locking phenomenon [15]. Among various approaches to elimination of locking, high-order approximation is a natural choice but it has been seldom seen in finite element formulations. The weak form quadrature element method (QEM) is an efficient numerical method wherein numerical integration is carried out before derivatives are approximated using differential quadrature analogs. Because of the feasibility of high-order approximation and the coincidence of integration points and nodes within an element, the QEM can use much fewer degrees-of-freedom, even a single element for a beam member, to tackle many problems with high accuracy. So far the QEM has shown remarkable superiority in beam-like structural analysis [16–20]. The successful application of the QEM to nonlinear analysis of geometrically exact shear-deformable beams demonstrates its effectiveness with no compromise in strain-objectivity [16,20].

In this paper, the QEM is applied to a total Lagrangian formulation of the shear-rigid beam model. The total Lagrangian formulation is favored over either the updated Lagrangian formulation or the Eulerian formulation in quadrature element analysis. This is mainly attributable to fact that the advantages of the unaltered reference configuration in the total Lagrangian formulation can be exploited to the maximum.

One of the aims of the present paper is to weigh the pros against the cons of the weak form quadrature element analysis of shear-rigid geometrically exact beams. For spatial shear-rigid beams, two rotation variables of a certain cross-section are expressed by translations on the centroidal axis for the shear-rigid condition to enforce the zero shear deformation constraint, while the drilling rotation variable remains independent. The

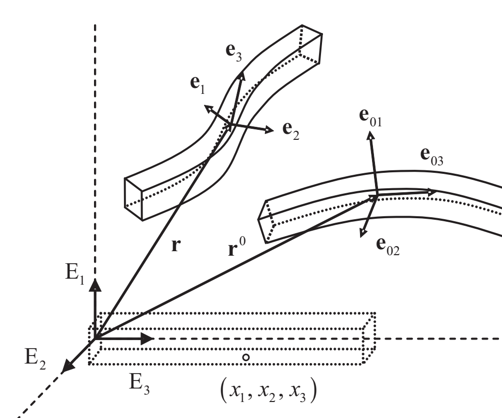

Fig. 1. Reference, initial and current configurations of beam.

major merits of the present approach include: (a) satisfaction of strain-objectivity without extra effort; (b) reduction of the number of degrees-of-freedom compared with shear-deformable beams; (c) circumvention of the so-called interdependent perplexity of translations and rotations of spatial shear-deformable beams [21].

The remaining portion of the present paper is organized as follows. Section 2 sums up spatial geometrically exact shear-rigid beam theory, followed by the weak form quadrature element formulation in Section 3. Section 4 presents three benchmark examples to demonstrate the effectiveness of the formulation. Conclusions are drawn in Section 5. Relatively lengthy explicit expression of the tangent stiffness matrix for the beam element is given in Appendix A. In addition, a proof of the strain-objectivity of the present formulation is furnished in Appendix B.

2. Shear-rigid geometrically exact beam theory

The beam model in the present paper allows for flexure, torsion and extension but neglects transverse shear deformation. Since no restriction is imposed on the magnitude of the displacements or rotations in the model, it is geometrically exact. In general, a reference configuration and a current configuration are needed for description of a spatial beam. As shown in Fig. 1, the reference configuration of the beam is assumed to have straight centroidal axis and a right-handed orthonormal frame $\{\boldsymbol{E}_1,\ \boldsymbol{E}_2,\ \boldsymbol{E}_3\}$ serving as the three base vectors for the Cartesian coordinate system is introduced. The origin of the frame is located on the centroidal axis of the reference beam. The frame is attached to the beam cross-section and its axes oriented along the principal axes of inertia. The cross-section of the beam is assumed to be rigid and therefore the shape remains unchanged during deformation.

In the current configuration, a cross-section of the beam is determined by the position vector of its centroid, denoted by $\mathbf{r}$, and an orthonormal frame $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ where $\mathbf{e}_3$ points to the normal direction of the cross-section and $\mathbf{e}_1$ and $\mathbf{e}_2$ are aligned with two principal axes of the cross-section inertia. In the case of initially curved beam, an initial configuration is usually needed. Similar to the description of the current configuration, position vector $\mathbf{r}_0$ and orthonormal frame $\{\mathbf{e}_{01},\ \mathbf{e}_{02},\ \mathbf{e}_{03}\}$ are used correspondingly to locate a cross-section in the initial configuration. Define s as the arc length parameter of the beam. The following vectors are introduced in the current and the initial configurations:

$$\mathbf{a}=\frac{\partial \mathbf{r}}{\partial s};\quad \mathbf{b}\times\mathbf{e}_i=\frac{\partial \mathbf{e}_i}{\partial s}$$
$$\mathbf{a}_0=\frac{\partial \mathbf{r}^0}{\partial s};\quad \mathbf{b}_0\times\mathbf{e}_{0i}=\frac{\partial \mathbf{e}_{0i}}{\partial s} \tag{1}$$

which are related to the strain measures of the beam by

$$\begin{aligned}
&\mathbf{a}=\gamma_{z1}\mathbf{e}_1+\gamma_{z2}\mathbf{e}_2+(\varepsilon_z+1)\mathbf{e}_3;\\
&\mathbf{b}=\kappa_{z1}\mathbf{e}_1+\kappa_{z2}\mathbf{e}_2+\gamma_{z3}\mathbf{e}_3;\\
&\mathbf{a}_0=\gamma_{01}\mathbf{e}_{01}+\gamma_{02}\mathbf{e}_{02}+(\varepsilon_0+1)\mathbf{e}_{03};\\
&\mathbf{b}_0=\kappa_{01}\mathbf{e}_{01}+\kappa_{02}\mathbf{e}_{02}+\gamma_{03}\mathbf{e}_{03}.
\end{aligned} \tag{2}$$

The strain measures are then extracted from Eq. (2), leading to transverse shear strains of the cross section $(\gamma_{z1}-\gamma_{01})$ and $(\gamma_{z2}-\gamma_{02})$, bending strains $(\kappa_{z1}-\kappa_{01})$ and $(\kappa_{z2}-\kappa_{02})$, and axial and torsion strain $(\varepsilon_z-\varepsilon_0)$ and $(\gamma_{z3}-\gamma_{03})$, respectively. The strain vectors for a spatial beam are defined as

$$\boldsymbol{\gamma}=\boldsymbol{\gamma}_z-\boldsymbol{\gamma}_0=\begin{pmatrix}\gamma_{z1}-\gamma_{01} & \gamma_{z2}-\gamma_{02} & \varepsilon_z-\varepsilon_0\end{pmatrix}^T;$$
$$\boldsymbol{\kappa}=\boldsymbol{\kappa}_z-\boldsymbol{\kappa}_0=\begin{pmatrix}\kappa_{z1}-\kappa_{01} & \kappa_{z2}-\kappa_{02} & \gamma_{z3}-\gamma_{03}\end{pmatrix}^T \tag{3}$$

In the shear-rigid hypothesis of a beam, the tangent of the centroidal axis of the beam is perpendicular to the cross-section,