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Adaptive path following schemes for problems with softening

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ABSTRACT

The present contribution deals with the question how structures with softening material behavior can be controlled in a numerical analysis beyond limit points, when conventional path following schemes fail. For nonlinear problems with localized cracks, adaptive path following schemes that increase numerical robustness, minimize user interference and avoid nonphysical (artificial) unloading are presented. In the methods proposed, a control region is identified where control parameters are evaluated. This control region adapts with the continuation of the crack tip. Robustness and applicability of the schemes are illustrated by numerical examples.

1. Introduction

The mechanical response of statically loaded structures until failure is governed by external loads, prescribed deformations or environmental effects, just to mention the most frequent 'loading' situations. Typically large structures are load controlled such as by gravity and life load, whereas small structural elements within a structure depend on the deformations of the neighbored structural parts; if they are less stiff than the main structure their contribution to the overall load carrying capacity diminishes during further loading of the entire structure. Depending on the controlling parameter, i.e. load or displacement, the load carrying capacity ceases at certain limit points leading to dynamic snap-through or snap-back phenomena eventually leading to complete structural failure.

These typical phenomena are reflected in related experiments. Larger specimens may be loaded by increasing external loads. However, structural elements are mostly controlled by selected kinematic variables like characteristic displacements or strains; examples are the elongation in a tension test or the crack opening displacement (COD) control experiment; see for example van Mier [1] for concrete. Other controlling mechanisms exist, e.g. pressure or volume control for vessels.

The situation of selecting proper controlling parameters in experiments as well as in numerical analyses is similar for geometrically and materially nonlinear response or its combination. However, for path dependent materials exhibiting damage or plasticity it becomes much more delicate due to potential local unloading. Although this is in certain cases a real physical phenomenon it may happen in conventional path following schemes like the arc-length method as a nonphysical artifact.

The present contribution deals with the question how these phenomena in the post-critical regime, i.e. beyond limit points, can be controlled in a numerical analysis. We concentrate on the determination of static equilibrium paths since they are a key to predictability of these states. They contain information about stability, imperfection sensitivity, ductility and robustness as well as potential dynamic phenomena of the corresponding structure. A variety of control types can be applied for tracing static nonlinear equilibrium paths. The choice of an appropriate constraint equation for the incremental iterative scheme is crucial and affects the convergence properties decisively. For nonlinear problems, advanced procedures like the *constant arc-length method* have been introduced by Riks [2] and Wempner [3]. Crisfield [4] proposed a quadratic spherical or cylindrical constraint equation, Ramm [5] presented the *updated normal method*, iterating on an updated normal plane. Schweizerhof and Wriggers [6] proposed a consistent linearization of the constraint equation to circumvent selecting the proper root for the quadratic problem. Over the last few decades, these methods have been applied for geometrically, as well as for materially nonlinear problems. Several modifications have been introduced in the mean time, see e.g. Wriggers [7] or Geers [8], and also have been implemented in commercial codes.

Most of these schemes can handle smooth geometrically nonlinear problems fairly well, but still have problems for path dependent material, in particular in the post-critical range. In this case strain localization and artificial unloading introduce additional challenges,

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these methods cannot always handle. It is known from the literature that global norms, appearing in the methods mentioned above, are inappropriate for many materially nonlinear problems. Geometric nonlinearity is taken into account in the numerical examples. However, particular aspects of geometric nonlinearity, like buckling, bifurcation and instabilities, are not in the focus of this paper. Dealing with this class of problems, de Borst [9] presented the *indirect displacement control*, for example using the crack mouth opening displacement as a controlling parameter. This method offers a robust solution technique, but requires a priori knowledge of the structural behavior in order to assign the controlling degrees of freedom before computation. Chen and Schreyer [10] prescribe selected strain components in the most critical point of the body. This *total strain control* necessitates continuously growing strains to gain monotonic behavior of the constraint equation. Unloading of the maximally strained element may lead to an algorithmic failure. As a generalization of these approaches Geers [8,11] proposes an adaptive scheme, the *subplane control approach*, that uses incremental predictor solutions to select or weigh control parameters for the subsequent iteration procedure. His approach, prescribing incremental changes instead of total values, does not require continuously growing control quantities. Rots et al. [12] introduce a sequentially linear analysis for concrete and masonry structures and apply a secant approximation replacing the softening branch by a saw-tooth approximation. In the proposal of Lorentz and Badel [13], the choice of the scalar constraint equation is based on the maximum value of the elastic predictor of the yield function. Another proposal of [13] is a normalized prescribed strain at the Gauss point exhibiting the maximum strain increment. Gutiérrez' *energy release control* [14] is a dissipation based arc-length control. In [15] it is extended to geometrically nonlinear damage and geometrically linear plasiticity. This method is well suited for dissipative parts of the path as the rate of dissipation is positive. For problems that exhibit elastic loading during the continuation of the equilibrium path, the constraint equation needs to be switched to a different control type which does not rely on dissipation. The energy release control is an elegant and consistent method. The authors hoped to get smooth transient regions of dissipative and elastic path sections by controlling a quantity implying elastic and dissipative changes. Furthermore, these transient areas should be overcome with a small number of load steps.

It can be stated that several situations exist where the control equation may fail, for example at limit points or sharp snap-backs. Furthermore, artificial unloading can occur for path dependent material.

1.1. Objective

For the present study the typical localized failure mechanism of an evolving individual crack based on a continuum damage model with softening is selected as a benchmark for a path following scheme. Additionally, one example without prescribed damage zone is used as a test example for the methods presented.

Without loss of generality, isotropic damage with linear softening is selected; an element related fracture energy based regularization is applied. Geometrical nonlinearities are taken into account.

After the basic equations for the path following schemes are summarized, the mentioned issue of artificial unloading is picked out as a central theme. In view of the mentioned complex of problems we concentrate on adaptive schemes that increase numerical robustness, minimize user interference and avoid nonphysical (artificial) unloading. Since the adaptive arc-length method either in the complete or selected displacement version cannot avoid artificial unloading the concept of an Adaptive Strain Control (ASC) is pursued. For this a control region in the most active process zone monitoring the damage related equivalent strain is picked. This leads to the selection of a control parameter which constitutes the constraint equation. By this artificial unloading can be avoided. This is underlined by several numerical examples.

2. Basic equations

2.1. Path following scheme

For the nonlinear problems studied in this paper proportional loading with a load factor λ is assumed. The problem contains n unknown degrees of freedom. Thus, the number of unknowns in a static incremental iterative scheme amounts to $n+1$: n degrees of freedom $\boldsymbol{D}$ and a load factor λ. Therefore in addition to the n equations equilibrating the internal and external forces $\mathbf{F}_{\text{int}}$ and $\mathbf{F}_{\text{ext}}$, respectively

$$\mathbf{R} = \mathbf{F}_{\text{int}}(\mathbf{D},\lambda) - \mathbf{F}_{\text{ext}}(\mathbf{D},\lambda) = \mathbf{0} \tag{1}$$

a scalar constraint equation is required. In the following it will also be referred to as the control equation

$$f = c - \hat{c} = 0 \tag{2}$$

Each incremental step allows choosing a new constraint equation. Thus, for m steps, up to m constraint equations are used during computation. The constraint or control equation filters equilibrium points out of the infinite number of equilibrium points of an equilibrium path. The points displayed satisfy prescribed selection criteria. Displacement control, for instance, $f = \Delta D_c - \Delta\hat{D}$, selects points in a fixed distance and direction from the current equilibrium point with the prescribed displacement $\Delta\hat{D}$. $f = \Delta\lambda - \Delta\hat{\lambda}$ represents a constraint equation that increments the load factor λ. A popular control function is the arc-length control, $f = \Delta s - \Delta\hat{s}$. The arc-length s is not a physical but a purely numerical quantity, which may be geometrically interpreted as the arc-length of the equilibrium path.

In the control procedures presented in this paper, the predictor in the first increment is the scaled tangent to the nonlinear problem. In the remaining steps, the secant from the previous to the current equilibrium point is used as a predictor for the next step. Thus, the constraint equation is satisfied in every predictor step. A Newton–Raphson scheme is applied for corrector iterations. Equilibrium equations and the constraint equation are linearized as proposed by Schweizerhof and Wriggers [6].

In the following, incremental changes will be marked with Δ, iterative changes will be described with δ. Note that δ is not meant to indicate variations in this paper:

$$\begin{aligned} \text{LIN}\ \mathbf{R} &= \mathbf{R}(\mathbf{D}^i,\lambda^i) + \frac{\partial \mathbf{R}(\mathbf{D}^i,\lambda^i)}{\partial \mathbf{D}^i}\delta\mathbf{D}^{i+1} + \frac{\partial \mathbf{R}(\mathbf{D}^i,\lambda^i)}{\partial \lambda^i}\delta\lambda^{i+1} \\ \text{LIN}\ f &= f(\mathbf{D}^i,\lambda^i) + \frac{\partial f(\mathbf{D}^i,\lambda^i)}{\partial \mathbf{D}^i}\delta\mathbf{D}^{i+1} + \frac{\partial f(\mathbf{D}^i,\lambda^i)}{\partial \lambda^i}\delta\lambda^{i+1} \end{aligned} \tag{3}$$

The derivative of f with respect to load factor λ is $\partial f/\partial\lambda = \partial f/\partial\mathbf{D}\cdot\partial\mathbf{D}/\partial\lambda$. i is the iteration index. Note that a derivative with respect to absolute values is exactly the same as deriving with respect to incremental changes. For prescribed displacements, e.g. for a permanent settlement, the displacements $\mathbf{D}$ depend on λ, then playing the role of an incrementation factor. In matrix representation

$$\begin{bmatrix} \mathbf{K}_\text{T} & \mathbf{R}_{,\lambda} \\ f_{,\mathbf{D}} & f_{,\lambda} \end{bmatrix} \begin{bmatrix} \delta\mathbf{D} \\ \delta\lambda \end{bmatrix} = -\begin{bmatrix} \mathbf{R} \\ f \end{bmatrix} \tag{4}$$

where $\mathbf{K}_\text{T} = \partial\mathbf{R}/\partial\mathbf{D}$ represents the tangent stiffness matrix, it can be noticed that a nonsymmetric system has to be solved. This may be avoided by applying the two step solution analogous to a procedure advocated by Crisfield [4] and Ramm [5], a technique