Available online at www.sciencedirect.com
ScienceDirect
Finite Elements in Analysis and Design 43 (2007) 901 – 911
FINITE ELEMENTS IN ANALYSIS AND DESIGN
www.elsevier.com/locate/finel

Graph coloration and group theory in dynamic analysis of symmetric finite element models

A. Kaveh*, H. Fazli

Centre of Excellence for Fundamental Studies in Structural Engineering, Iran University of Science and Technology, Narmak, Tehran 16, Iran

Received 8 February 2007; received in revised form 30 May 2007; accepted 7 June 2007
Available online 19 July 2007

Abstract

In this article the idea of graph coloration from spectral graph theory is employed in conjunction with group theoretical concepts for efficient eigensolution of adjacency matrices of graphs. The application of the method is extended to free vibration analysis of symmetric finite element problems, constructing a graph model of the problem in a local symmetry adapted coordinate system.

Keywords: Graph coloration; Group theory; Finite elements; Symmetry; Eigenvalues; Eigenvectors

1. Introduction

Symmetry has been widely studied in structural and solid mechanics. Problems exhibiting symmetry properties encompass bifurcation, vibration and finite element formulation. Different forms of symmetry have been studied in [1–4]. Graph methods are employed in different eigenproblems in structural mechanics [5,6].

Group theory is a systematic means to exploit geometric symmetry. Group theory methods are applied to stability analysis and bifurcation problems [7–11], vibration analysis [12–14], various eigenproblems in structural mechanics [15–18], and formulation of stiffness and mass matrices for finite elements [19–21]. For a detailed discussion of general problems in the field of solid and structural mechanics one can refer to the excellent review paper of Ref. [22].

All these group theoretical approaches are based on a mathematical principle that the solution space is to be partitioned into a series of orthogonal subspaces. Using this principle, stiffness and mass matrices are put into block-diagonal forms through a transformation matrix. Then, the analysis of the whole system is reduced to analysis of smaller blocks, reducing memory requirement and lowering the computational effort.

In this article, combining group theoretical concepts with a concept from spectral graph theory, a new method is developed for factorization of the matrices connected to dynamic systems of finite elements. This method is mainly used for factoring characteristic polynomial of the adjacency matrix of a symmetric graph. The method is elaborated through multiple examples and an application to vibrating mass-spring systems in [23].

In this method stiffness and mass matrices are formulated in a symmetry adapted local coordinate system. Then a special graph is constructed for the symmetric problem, and the mass and stiffness matrices are factorized with regard to different colorations of this graph. These factors are obtained through some additions and subtractions of columns and rows of the matrix. The eigenvalues of general problem is the union of those of smaller factored problems. The eigenvectors computed for each factored problem are simply augmented with regard to corresponding coloration to obtain the eigenvectors of general problem.

The main advantage of the present method, compared to the group theoretical approaches is that the need for computation of a transformation matrix is eliminated and symmetry properties are obtained much easier through different colorations.

* Corresponding author. Tel.: +98 21 4202710; fax: +98 21 7454053.
E-mail address: alikaveh@iust.ac.ir (A. Kaveh).

0168-874X/$ - see front matter
doi:10.1016/j.finel.2007.06.002

Group theory is used as a guide and a controlling means for selecting subgroups which are exploited in colorations and for determining multiplicity of each factor.

2. Basic definitions and concepts

2.1. Basic definitions from graph theory

A *graph*, G, consists of a set $V(G)$ of elements called *nodes* (vertices or points) and a set $E(G)$ of elements called *members* (edges or arcs) together with a relation of incidence which associated with each member a pair of nodes, called its *ends*. A graph having two or more members joining the same pair of nodes is known as multi-graph, if the members have directions, the graph is called *directed graph* or di-graph. The graph may have a loop which is a member joining a node to itself. A di-graph which has weights on its members is called *weighted directed graph* or wd-graph.

In this paper, the weights are shown on the members except for the case of unit weights, which are omitted for clarity. A member with no direction sign is two sided of equal weights.

2.2. The concept of coloration

Let G be a wd-graph and $V_1 \cup V_2 \cup \cdots \cup V_k$ be a partition of its vertex set, such that each $v \in V_i$ is colored with color C_i. Let $\boldsymbol{D} = (d_{ij})$ be a square matrix of order k. We say that G has a $\boldsymbol{D}$-feasible coloration if for each $i, j = 1, 2, \ldots, k$, the sum of the weights of the members issuing from any vertex $v \in V_i$, and terminating in the vertices of V_j is d_{ij}. In other words, if a_{vu} be the weight of directed member from v to u, then

$$\sum_{u \in V_j} a_{vu} = d_{ij}, \quad v \in V_i. \tag{1}$$

The partition $V_1 \cup V_2 \cup \cdots \cup V_k$ is also known as equitable partition [24]. The matrix $\boldsymbol{D}$ as an adjacency matrix, specifies the wd-graph D, which is called a divisor graph of G [25].

3. A graph theory method of obtaining factors of a graph from its coloration

Consider the wd-graph G which has a $\boldsymbol{D}$-feasible coloration. Color the vertices of G with k colors. From each color class C_i choose a vertex as the representative and denote these k representative nodes with V_D and the remaining nodes by V_C. In G, for each vertex $v \in V_D$ and for each member issuing from v to vertex $u \in V_C$ with weight a_{vu}, add a directed member of weight a_{vu}, from v to vertex z in V_D, with the same color as u. Now, for each $u \in V_C$ and for each member entering u from any vertex $v \in V_D$, having weight a_{vu} add members of weight $-a_{vu}$, from all vertices of V_C having the same color as v, to vertex u. In G, delete all members between the sets V_D and V_C. The wd-graph formed on the set V_D is *divisor D*, and the one formed on V_C is called *co-divisor C*.

4. Coloration and the spectrum of a graph

Let $\mathbf{A}$ be the adjacency matrix of the graph G. The polynomial $P_G(\lambda) = |\lambda \boldsymbol{I} - \boldsymbol{A}|$ is called the characteristic polynomial of graph G. The set of eigenvalues of $\mathbf{A}$, i.e. the roots of $P_G(\lambda)$, is called the *ordinary spectrum* of G, denoted by $\boldsymbol{Sp}(G) = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ [26]. The following theorem establishes the relation between characteristic polynomial of a graph and that of a divisor obtained from a coloration of the graph.

Theorem (*Cvetkovic et al.* [25]). *Let D and G be arbitrary multi-di-graphs such that D is a divisor of G*; *then*, *the polynomial of D divides that of G*:

$$P_D(\lambda) | P_G(\lambda). \tag{2}$$

5. Graph coloration and symmetry

5.1. The automorphism group

A permutation p on the set of vertices of a graph G is called an *automorphism* of G if it is adjacency preserving [25, 26, Part 3]. Suppose that $\mathbf{P}$ is the matrix representation of p. If $\mathbf{A}$ is the adjacency matrix of G, p is an automorphism if and only if

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{A}. \tag{3}$$

The automorphisms of G form a group $\Gamma = \Gamma(G)$ (*gamma*) called the *automorphism group* of G.

This group has the following properties:

(1) For $p_1, p_2 \in \Gamma$, the composition $p_1 p_2$ (obtained by applying first p_1, then p_2) is in Γ.
(2) The identity map (e) (which fixes every point) belongs to Γ.
(3) If $p \in \Gamma$, then the inverse function p^{-1} belongs to Γ.

A *subgroup* of a group Γ is a subset $H$of Γ such that H equipped with the restriction of the operation of Γ is itself a group.

5.2. The symmetry group

Since we are concerned with symmetric graphs and representing the symmetry in graphs is more convenient and tangible using the symmetry operations (especially rotations), hence the automorphisms of a graph will be related to the rotational elements of the symmetry group. Therefore, anywhere in this paper that a rotation is addressed, in fact a permutation of the vertices of the graph is intended.

The symmetry of a graph is described by the set of all those transformations which brings the graph into coincidence with itself. Any such transformation is called a *symmetry transformation* and the set of all such transformations forms a group, known as the *symmetry group* of the graph.