



Singular values of two-parameter families $\lambda((b^z - 1)/z)^\mu$ and $\lambda(z/(b^z - 1))^\eta$

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Abstract

The singular values of two kinds of two-parameter families of functions (i) $f_{\lambda,\mu}(z) = \lambda((b^z - 1)/z)^\mu$ and $f_{\lambda,\mu}(0) = \lambda(\ln b)^\mu$, $\mu > 0$, (ii) $g_{\lambda,\eta}(z) = \lambda(z/(b^z - 1))^\eta$ and $g_{\lambda,\eta}(0) = \lambda/(\ln b)^\eta$, $\eta > 0$; $\lambda \in \mathbb{R} \setminus \{0\}$, $z \in \mathbb{C}$, $b > 0$, $b \neq 1$ are described. It is shown that all the critical values of $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ lie interior and exterior of the disk centered at origin and having radii $|\lambda(\ln b)^\mu|$ and $|\lambda/(\ln b)^\eta|$ respectively. Further, it is proved that both the functions $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ have infinitely many singular values for all $b > 0$, $b \neq 1$. © 2016 The Author. Production and hosting by Elsevier B.V. on behalf of Taibah University. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

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1. Introduction

Usually, it is crucial to study the dynamical properties of entire or meromorphic functions if singular values exist. The importance of singular values in the dynamics of transcendental functions can be seen in [1–4]. The dynamics of one parameter family λe^z , that has only one singular value, was vastly explored in [5,6]. This exponential family is simpler than other families which have more than one or infinitely many singular values. Some dynamical properties of such types of families of functions including exponential map were

studied in [7–11]. Often these investigations are very applicable for studying of iterative methods associate to entire and meromorphic transcendental functions which have either critical values or asymptotic values, or both. Enormously, singular values are useful in the dynamics of entire and meromorphic transcendental functions for describing the Julia sets and the Fatou sets [8–12].

The present paper devotes to investigate the singular values of two kinds of two-parameter families of transcendental functions. For this purpose, we consider the following two-parameter families of transcendental entire and meromorphic functions respectively which are neither even nor odd and not periodic:

$$\mathcal{E} = \left\{ f_{\lambda,\mu}(z) = \lambda \left(\frac{b^z - 1}{z} \right)^\mu \quad \text{and} \right. \\ f_{\lambda,\mu}(0) = \lambda(\ln b)^\mu : \mu > 0, \lambda \in \mathbb{R} \setminus \{0\}, \\ \left. z \in \mathbb{C}, b > 0, b \neq 1 \right\}$$

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$$\mathcal{M} = \left\{ g_{\lambda,\eta}(z) = \lambda \left(\frac{z}{b^z - 1} \right)^\eta \quad \text{and} \right. \\ \left. g_{\lambda,\eta}(0) = \frac{\lambda}{(\ln b)^\eta} : \eta > 0, \lambda \in \mathbb{R} \setminus \{0\}, \right. \\ \left. z \in \mathbb{C}, b > 0, b \neq 1 \right\}$$

The family $\mathcal{E}$ is a generalization of one parameter families of entire functions $\lambda((e^z - 1)/z)$ from [13], $\lambda((b^z - 1)/z)$ from [14] and $\lambda((e^z - 1)/z)^m$ from [15]; and the family $\mathcal{M}$ is a generalization of one parameter families of meromorphic functions $\lambda(z/(e^z - 1))$ from [16], $\lambda(z/(b^z - 1))$ from [17] and $\lambda(z/(e^z - 1))^m$ from [18]. Moreover, the families $\mathcal{E}$ and $\mathcal{M}$ based upon the positive real number b are also related to well known generalized (i) Bernoulli's generating function $(z/(e^z - 1))^\alpha e^{tz} = \sum_{k=0}^{\infty} B_k^{(\alpha)}(t)(z^k/k!)$ by choosing α nonzero real and $t = 0$ (ii) Apostol-Bernoulli's generating function $(z/(\lambda e^z - 1))^\alpha e^{tz} = \sum_{k=0}^{\infty} B_k^{(\alpha)}(t; \lambda)(z^k/k!)$ by choosing α nonzero real, $\lambda = 1$ and $t = 0$.

A point z^* is said to be a critical point of $f(z)$ if $f'(z^*) = 0$. The value $f(z^*)$ corresponding to a critical point z^* is called a critical value of $f(z)$. A point $w \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is said to be an asymptotic value for $f(z)$, if there exists a continuous curve $\gamma : [0, \infty) \rightarrow \hat{\mathbb{C}}$ satisfying $\lim_{t \rightarrow \infty} \gamma(t) = \infty$ and $\lim_{t \rightarrow \infty} f(\gamma(t)) = w$. A singular value of f is defined to be either a critical value or an asymptotic value of f .

In Theorem 2.1, it is shown that both the functions $f'_{\lambda,\mu}(z)$ and $g'_{\lambda,\eta}(z)$ have no zeros in the left half plane for $0 < b < 1$ and in the right half plane for $b > 1$. It is found that, in Theorem 2.2, the functions $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ map (i) the right half plane interior and exterior of the disk for $0 < b < 1$ respectively (ii) the left half plane interior and exterior of the disk for $b > 1$ respectively. In Theorem 2.3, it is seen that all the critical values lie interior and exterior of the disk centered at origin and having radii $|\lambda(\ln b)^\mu|$ and $|\lambda/(\ln b)^\eta|$ respectively. Further, in Theorem 2.4, it is proved that the functions $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ have infinitely many singular values for all $b > 0, b \neq 1$.

2. Singular values of $f_{\lambda,\mu} \in \mathcal{E}$ and $g_{\lambda,\eta} \in \mathcal{M}$

Let $D_r(0)$ be a disk centered at origin and radius r . Suppose that the right half and left half planes are given by $H^+ = \{z \in \hat{\mathbb{C}} : \operatorname{Re}(z) > 0\}$ and $H^- = \{z \in \hat{\mathbb{C}} : \operatorname{Re}(z) < 0\}$ respectively. In the following theorem, it is shown that both $f'_{\lambda,\mu}(z)$ and $g'_{\lambda,\eta}(z)$ have no zeros in the left half plane for $0 < b < 1$ and the right half plane for $b > 1$:

Theorem 2.1. Let $f_{\lambda,\mu} \in \mathcal{E}$ and $g_{\lambda,\eta} \in \mathcal{M}$. Then,

- for $0 < b < 1$, $f'_{\lambda,\mu}(z)$ and $g'_{\lambda,\eta}(z)$ have no zeros in the left half plane H^- .
- for $b > 1$, $f'_{\lambda,\mu}(z)$ and $g'_{\lambda,\eta}(z)$ have no zeros in the right half plane H^+ .

Proof.

- For $z \neq 0$, $f'_{\lambda,\mu}(z) = \lambda \mu \left(\frac{b^z - 1}{z} \right)^{\mu-1} \frac{(z \ln b - 1)b^z + 1}{z^2}$ and $f'_{\lambda,\mu}(0) = \lambda \mu \frac{(\ln b)^{\mu+1}}{2}$, the zeros of $f'_{\lambda,\mu}(z)$ are given by $b^{-z} = 1 - z \ln b$ and $z = 2\pi i / \ln b$, where p is nonzero integer.

For $z \neq 0$, $g'_{\lambda,\eta}(z) = \lambda \eta \left(\frac{z}{b^z - 1} \right)^{\eta-1} \frac{(1 - z \ln b)b^z - 1}{(b^z - 1)^2}$ and $g'_{\lambda,\eta}(0) = -\lambda \eta \frac{1}{2(\ln b)^{\eta-1}}$, the zeros of $g'_{\lambda,\eta}(z)$ are given by $b^{-z} = 1 - z \ln b$.

It is found in [14] that the equation $b^{-z} = 1 - z \ln b$ has no any solution in H^- . Therefore, it shows that the function $f'_{\lambda,\mu}(z)$ and $g'_{\lambda,\eta}(z)$ have no zeros in H^- for $0 < b < 1$.

- The proof of this part is similar as part (a).

□

The following theorem proves that the functions $f_{\lambda,\mu} \in \mathcal{E}$ and $g_{\lambda,\eta} \in \mathcal{M}$ map the right and the left half planes interior and exterior of the disk respectively:

Theorem 2.2. Let $f_{\lambda,\mu} \in \mathcal{E}$ and $g_{\lambda,\eta} \in \mathcal{M}$. Then,

- for $0 < b < 1$, $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ map the right half plane H^+ interior of $D_{|\lambda(\ln b)^\mu|}(0)$ and exterior of $D_{|\lambda/(\ln b)^\eta|}(0)$ respectively.
- for $b > 1$, $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ map the left half plane H^- interior of $D_{|\lambda|(\ln b)^\mu}(0)$ and exterior of $D_{|\lambda|/(\ln b)^\eta}(0)$ respectively.

Proof.

- For $0 < b < 1$, suppose that the line segment γ is defined by $\gamma(t) = tz$, $t \in [0, 1]$. Further, let the function $h(z) = b^z$ for an arbitrary fixed $z \in \mathbb{C}$. Since $M \equiv \max_{t \in [0, 1]} |h(\gamma(t))| = \max_{t \in [0, 1]} |b^{tz}| < 1$ for $z \in H^+$, then

$$\int_{\gamma} h(z) dz = \int_0^1 h(\gamma(t)) \gamma'(t) dt = z \int_0^1 b^{tz} dt = \frac{1}{\ln b} (b^z - 1) \\ |b^z - 1| = |\ln b| \left| \int_{\gamma} h(z) dz \right| \leq M |z| |\ln b| < |z| |\ln b| \quad (1)$$

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