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The use of Timoshenko's exact solution for a cantilever beam in adaptive analysis

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Abstract

The exact solution for the deflection and stresses in an end-loaded cantilever is widely used to demonstrate the capabilities of adaptive procedures, in finite elements, meshless methods and other numerical techniques. In many cases, however, the boundary conditions necessary to match the exact solution are not followed. Attempts to draw conclusions as to the effectivity of adaptive procedures is therefore compromised. In fact, the exact solution is unsuitable as a test problem for adaptive procedures as the perfect refined mesh is uniform. In this paper we discuss this problem, highlighting some errors that arise if boundary conditions are not matched exactly to the exact solution, and make comparisons with a more realistic model of a cantilever. Implications for code verification are also discussed.

Keywords: Adaptivity; Finite element method; Meshless; Closed form solution; Beam; Error estimation; Meshfree

1. Introduction

Adaptive methods are well-established for analysis of elastostatic problems using finite elements and are now emerging for meshless methods. Many publications in this area measure the capability of adaptive procedures by comparison with the limited number of exact solutions which exist. One of these problems is that of a cantilever subjected to end loading [1]. The purpose of this paper is to highlight potential sources of error in the use of this solution relating to the particular boundary conditions assumed and to show that it is a solution neither appropriate for testing adaptivity nor as a model of a real cantilever.

While some may consider that the observations we make are self-evident and well-known, the literature contains many counter examples. This paper provides graphic illustration of the effect of various boundary conditions on the cantilever beam solution. To our knowledge these effects have not been presented in detail in the existing literature. We also demonstrate the difference between the behaviour of a real cantilever and the idealised Timoshenko cantilever. It is our hope that this paper will help to reduce the misuse of the Timoshenko cantilever beam in the evaluation of adaptive analysis schemes, and perhaps encourage the use of a more realistic cantilever beam model as a benchmark problem instead.

2. Problem definition

Fig. 1 shows a cantilever beam of depth D, length L and unit thickness, which is fully fixed to a support at $x = 0$ and carries an end load P. Timoshenko and Goodier [1] show that the stress field in the cantilever is given by

$$\sigma_{xx} = \frac{P(L-x)y}{I}, \tag{1}$$

$$\sigma_{yy} = 0, \tag{2}$$

$$\tau_{xy} = -\frac{P}{2I}\left[\frac{D^2}{4} - y^2\right] \tag{3}$$

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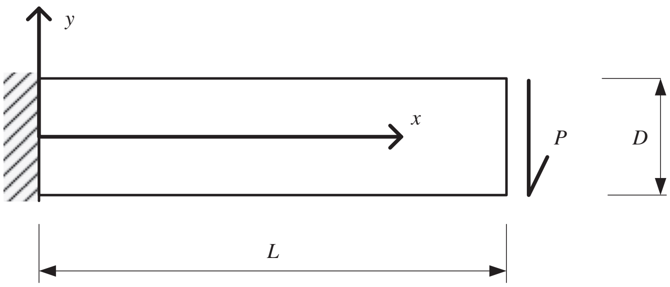

Fig. 1. Coordinate system for the cantilever problem.

and the displacement field $\{u_x, u_y\}$ is given by

$$u_x = -\frac{Py}{6EI}\left[(6L-3x)x + (2+v)\left[y^2 - \frac{D^2}{4}\right]\right], \tag{4}$$

$$u_y = -\frac{P}{6EI}\left[3vy^2(L-x)+(4+5v)\frac{D^2x}{4}+(3L-x)x^2\right], \tag{5}$$

where E is Young's modulus, v is Poisson's ratio and I is the second moment of area of the cross-section.

Crucially [1] states that "... it should be noted that this solution represents an exact solution only if the shearing forces on the ends are distributed according to the same parabolic law as the shearing stress τ_{xy} and the intensity of the normal forces at the built-in end is proportional to y."

If this is ignored then the solution given by Eqs. (1)–(5) is incorrect for the ends of the cantilever.

The solution has been widely used to demonstrate adaptive procedures in finite element methods (e.g. [2–4]), boundary elements (e.g. [5]) and (most commonly) meshless methods (e.g. [6–12]). However, inspection of Eqs. (1)–(5) shows the stresses to be smooth functions of position, with no stress concentrations or singularities. Therefore, it would not appear to be a suitable test for an adaptive procedure where a uniform mesh or grid is refined to improve accuracy locally to areas of high gradients in field quantities. Any analysis that yields a non-smooth field for this problem (and there are many examples in the literature on adaptivity) is an analysis of a cantilever under different boundary conditions, for which the exact solution is incorrect.

The performance of an adaptive procedure is widely measured using the effectivity index θ which is defined for a refined mesh (or grid) as

$$\theta = \frac{\eta^*}{\eta}, \tag{6}$$

where η is the error estimate based on the difference between the solution from the fine mesh the coarse mesh, and η^* is the error estimate based on the difference between the exact solution and the coarse mesh [2]. The effectivity index θ for the cantilever problem is meaningless unless the boundary conditions are modelled as specified in [1].

3. Analysis of the Timoshenko and Goodier cantilever

It is not possible to model the cantilever in [1] using finite elements by applying the stated traction boundary conditions only. In that case the problem is unstable as there is an unrestrained rotational rigid-body mode. Instead stability and an accurate model can be achieved by imposing the load as a parabolically varying shear force at each end according to Eq. (3) and by applying essential boundary conditions at the "fixed end" according to Eqs. (4) and (5).

To demonstrate the effects of using different boundary conditions five adaptive analyses of cantilevers have been carried out. The boundary conditions for each analysis are shown in Fig. 2 and have been chosen to match the conditions used in various previous publications. In analysis A full-fixity is applied to the nodes at the support, while the load P is applied uniformly distributed over the vertical surface at $x = L$, e.g. Refs. [2,13]. In analysis B the load is instead distributed parabolically, e.g. [6]. In analysis C, fixity at the support is released via rollers above and below the fixed mid-point, e.g. [14–16]. In analysis D traction boundary conditions are applied at $x = 0$ to the cantilever of analysis C. Finally, analysis E includes parabolic variation of applied shear traction at $x = L$ with essential boundary

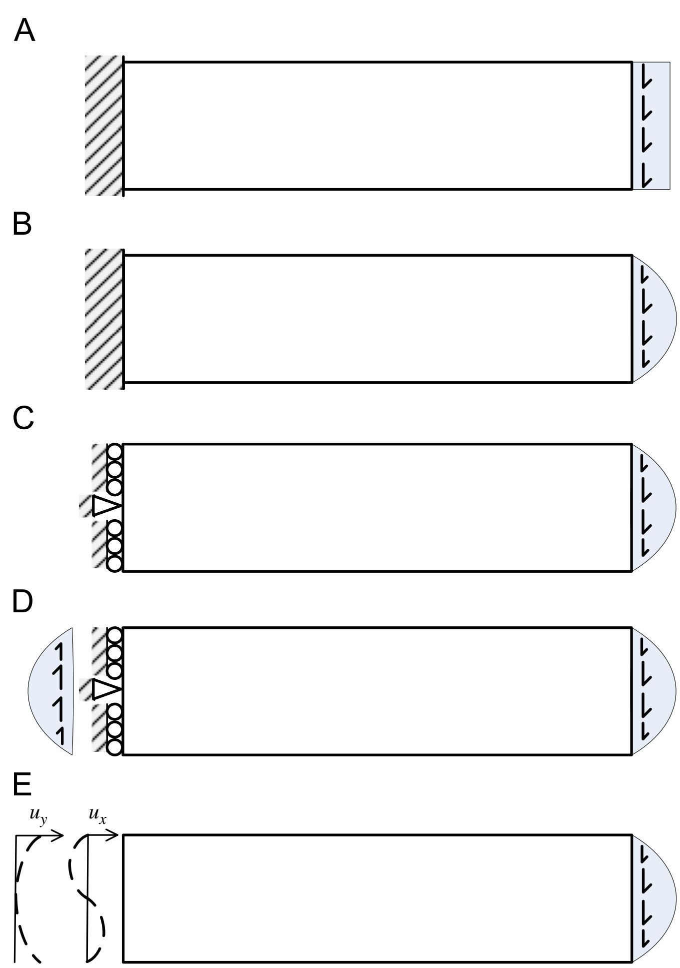

Fig. 2. The five different cantilever problems analysed.