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Two higher order hybrid-Trefftz elements for thin plate bending analysis

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ABSTRACT

To analyze thin bending plates, a triangular and a quadrilateral element are presented in this paper. These suggested elements are formulated based on the hybrid-Trefftz method. The triangular element is named as THT-15, and it has 6 nodes and 18 degrees of freedom. The quadrilateral element is composed of 8 nodes and 24 degrees of freedom, and it is denoted as QHT-23. Two independent fields are introduced: one within the element and the other on the edges of the element. The internal field satisfies the governing equation of the thin plates. The boundary field is related to the nodal degrees of freedom by shape functions. For better capability, the shape functions of a 3 node Euler–Bernoulli beam are used for each edge of the element. The order of these functions for the deflection, rotation and torsion fields is equal to five, four and two, respectively. By depicting these fields in general coordinates, *x*–*y*, shape functions for the elements' edges are obtained. Several numerical tests are performed to assess the robustness of the suggested elements. The findings demonstrate the high accuracy of the proposed elements in analyzing the thin bending plates.

1. Introduction

Thin plate bending is widely utilized in the fields of aerospace, mechanics and civil engineering. These kinds of plates are formulated based on Kirchhoff's theory. In this theory, the shear deformation of the element is not considered [1]. The finite element method has been extensively used to analyze thin bending plates. Consequently, various elements have been presented. Some of these elements are formulated by employing Ritz's strategy. In this approach, the element's degree of freedom and the field function's order are enlarged to increase the efficiency of the finite element scheme [2–8]. It is worth emphasizing that the accuracy of this method is not high. In general, continuity does not exist in inter-elements when the aforementioned tactic is deployed. Hybrid functionals are used to formulate some other types of finite elements. In these techniques, the number of master fields is more than one.

By introducing independent fields in the interfaces, a hybrid functional can be established. This functional is utilized to create finite elements with high efficiency. Several independent fields are deployed by the hybrid functional. It is worth emphasizing that this functional employs independent boundary displacement fields. Fortunately, the continuity condition is not required to be established in the element's boundaries [9]. Pian formulated the first hybrid element for plane stress problems [10]. Prior to using this method in analyzing bending plates, it was employed in solving plane problems. By employing the hybrid functional, extensive research has been performed in the area of solving the bending plates, so far [11–13].

The ability of the hybrid formulation in constructing efficient elements resulted in development of hybrid Trefftz (HT) methods. It should be added that independent displacement fields are applied in this technique. To solve various problems in solid mechanics, both Ritz and hybrid-Trefftz techniques can be used. In contrast to Ritz's algorithm, the problems' governing differential equations are set up by a field function when the hybrid-Trefftz approach is employed. In other words, both homogenous and particular solutions of the governing equations are deployed to formulate the finite element. By using independent boundary displacement field and variational principles, weak forms of the compatibility between elements are obtained in the hybrid-Trefftz method. Note that integration on the boundary of the elements is permissible in this approach. The mentioned characteristic is considered as the main advantage of this technique. Clearly, integration on the boundary is much easier than integration on the surface. In addition, polygon elements can be produced when integration on the boundary is possible. To express the superiority of this approach over the conventional ones, some of its advantages are mentioned below [14]:

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(1) Continuity exists in the hybrid tactic, and it is not necessary to set up continuity as an additional condition.
(2) Without employing a new complex formulation, it is possible to define numerous degrees of freedom on the boundary of the element.
(3) The element has high accuracy because of appropriate polynomials, which satisfy the governing equation.

The Hybrid-Trefftz formulation was first employed to assess the distorted meshes of the thin plates by Jirousek and Leon in 1970s [15]. This technique soon attracted the attention of other researchers due to its high accuracy and efficiency. The aforementioned approach has been deployed successfully to solve planar problems, thin plates, thick plates, shells and three-dimensional problems [16]. For analysis of thin bending plates, various triangular and quadrilateral elements have been formulated by Jirousek [15–18]. In the aforesaid elements, the *p*-version tactic has been used. By defining virtual nodes in the middle of the element's sides, a boundary displacement field was obtained for the formulation. This procedure was slightly complicated. Rezaiee-Pajand et al. proposed efficient triangular and quadrilateral elements for analysis of the thin plates. In the study, they utilized the interpolation functions of the Euler–Bernoulli Beam with two nodes [19]. In addition, Rezaiee-Pajand and Karkon employed interpolation functions of Timoshenko's beam for boundary fields to solve Reissner–Mindlin plates [20]. To analyze these plates, Petrolito presented a quadrilateral element composed of four nodes and a triangular element with three nodes. He used quadratic boundary displacement and linear boundary rotation field [21,22]. Note that higher-order elements, such as quadrilateral elements with eight nodes and triangular elements with six nodes, have high accuracy in coarse meshes. Due to this characteristic, they have received considerable attention. However, these elements have not been employed in the hybrid-Trefftz methods. It should be added that these elements are extensively deployed in other approaches [23–26].

In this paper, by employing the hybrid-Trefftz functional, higher-order triangular and quadrilateral elements are presented to analyze the thin bending plates; these elements are composed of six and eight nodes, respectively. The homogenous and particular solutions of the governing equations are deployed to establish the inner displacement field of the elements. It is worth emphasizing that the inner displacement fields are the same in all the elements, which are used for analysis of the thin bending plates. Moreover, boundary interpolation functions relate nodal degrees of freedom to the independent boundary displacement field. In this formulation, the shape functions, of a 3 node Euler–Bernoulli beam, are utilized. The orders of the shape functions for the deflection, rotation and torsion fields are equal to five, four and two, respectively. By depicting the aforesaid fields in general coordinates, *x*–*y*, the shape functions of the presented elements' edges are obtained. Finally, several numerical tests are performed to investigate the robustness of these elements. The findings prove that the suggested elements are able to calculate the stresses and strains with high level of accuracy. Furthermore, it will be clearly demonstrated that the authors' elements are insensitive to mesh distortion.

2. Hybrid-Trefftz functional

The Hybrid-Trefftz functional with independent inner displacement field (u) and boundary displacement field ($\tilde{u}$) has the following appearance [27]:

$$\Pi(u,\tilde{u}) = \int_\Omega W^u \, d\Omega - \int_\Omega b_i u_i \, d\Omega - \int_{\Gamma_T} \overline{T}_i \tilde{u}_i \, d\Gamma_t - \int_\Gamma T_i (u_i - \tilde{u}_i) \, d\Gamma \quad (1)$$

in which W^u, b_i, T_i and $\overline{T}_i$ denote strain energy density function, body force, tractions, and prescribed traction along boundary Γ_T, respectively. The inner displacement field (u) which is utilized in this equation should satisfy the governing equation. Moreover, $\tilde{u}$ provides compatibility and continuity between elements in the weak form. The inner displacement and tractions can be stated as follows:

$$\{u\} = \{u_p\} + \sum_{j=1}^{m} \{\phi_j\} c_j = \{u_p\} + [\Phi]\{c\} \quad (2)$$

$$\{T\} = \{T_p\} + [\Theta]\{c\} \quad (3)$$

It should be added that this equality is the sum of a homogenous and a non-homogenous part. In this relation, $\{u_p\}$ and $[\Phi]\{c\}$ are particular and homogenous solutions, respectively, of the governing equation. The traction resulting from $\{u_p\}$ is demonstrated by $\{T_p\}$, and $[\Theta]\{c\}$ denotes the tractions obtained from the homogenous solution of the differential equation. Furthermore, the boundary displacement field is related to the displacement interpolation functions as follows:

$$\{\tilde{u}\} = [\tilde{N}]\{\tilde{d}\} \quad (4)$$

In this equality, the nodal displacements and boundary interpolation fields are shown by $\{\tilde{d}\}$ and $[\tilde{N}]$, respectively. To obtain a stiffness matrix and nodal displacements of the element, the functional (1) should be stationary with respect to the independent fields. For this purpose, the internal strain energy can be written as follows:

$$U = \int_\Omega W^u \, d\Omega = \frac{1}{2} \int_\Omega \sigma^u_{ij} \varepsilon^u_{ij} \, d\Omega \quad (5)$$

By inserting this equality in functional (1), and using integration by parts, the first variation of this functional with respect to internal displacement can be obtained as follows:

$$\delta \Pi(u,\tilde{u})|_{\delta u} = -\int_\Omega (\sigma_{ij,j} + b_i) \delta u_i \, d\Omega + \int_\Gamma T_i \delta u_i \, d\Gamma - \int_\Gamma \delta T_i (u_i - \tilde{u}_i) \, d\Gamma - \int_\Gamma T_i \delta u_i \, d\Gamma \quad (6)$$

In the hybrid-Trefftz formulation, the internal displacement is selected such that it satisfies the governing equation. Therefore, the first term on the right-hand side of Eq. (6) is equal to zero. As a result, the previous relationship will change to the next shape:

$$\delta \Pi(u,\tilde{u})|_{\delta u} = -\int_\Gamma \delta T_i (u_i - \tilde{u}_i) \, d\Gamma = 0 \quad (7)$$

Furthermore, the first variation of functional (1), with respect to boundary displacement, will have the following form:

$$\delta \Pi(u,\tilde{u})|_{\delta \tilde{u}} = \int_\Gamma T_i \delta \tilde{u}_i \, d\Gamma - \int_{\Gamma_T} \overline{T}_i \delta \tilde{u}_i \, d\Gamma = 0 \quad (8)$$

Inserting Eqs. (2)–(4) into Eqs. (7) and (8) leads to the succeeding relations:

$$\delta\{c\}^T [\{h\} + [H]\{c\} - [G]\{d\}] = 0 \quad (9)$$

$$\delta\{d\}^T [\{g\} + [G]^T \{c\}] = 0 \quad (10)$$

By deploying relationship (9), $\{c\}$ can be computed as follows:

$$\{c\} = [H]^{-1}[G]\{d\} - [H]^{-1}\{h\} \quad (11)$$

The parameters employed in the Eqs. (9)–(11) are defined in the following form:

$$\{h\} = \int_\Gamma [\Theta]^T \{u_p\} d\Gamma \quad (12)$$

$$\{g\} = \int_\Gamma [\tilde{N}]^T \{T_p\} d\Gamma - \int_{\Gamma_T} [\tilde{N}]^T \{\overline{T}\} d\Gamma \quad (13)$$