Finite Elements in Analysis and Design 74 (2013) 22–29

Contents lists available at SciVerse ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.com/locate/finel

# A finite element model for the bending and vibration of nanoscale plates with surface effect

K.F. Wang, B.L. Wang*

*Graduate School at Shenzhen, Harbin Institute of Technology, Harbin 150001, PR China*

ARTICLE INFO

*Article history:*
Received 19 December 2012
Received in revised form
10 April 2013
Accepted 11 May 2013
Available online 21 June 2013

*Keywords:*
Nanoscale plate
Finite element method
Surface residual stress
Surface elasticity

ABSTRACT

A continuum finite element model for the nanoscale plates considering the surface effect of the material is developed. Governing equations for Kirchoff and Mindlin nanoplates are derived by using the Galerkin finite element technique. The model is verified by comparing the results with available analytical solutions. The results indicate that, depending on the boundary conditions, the deflections and frequencies of the plate have a dramatic dependence on the residual surface stress and surface elasticity of the plates. The present model is an efficient tool for the analysis of the static and dynamic mechanical behaviors of nanoscale plates with complex geometry, boundary and loading conditions and material properties.



## 1. Introduction

Nanoscale plates are key components of actuators and sensors for nano-electromechanical system (NEMS) [1]. Naturally, it is importance to understand the static and dynamic mechanical behaviors of these advanced materials and structures for the design and manufacture of NEMS. Due to their high surface-to-volume ratio, structures at nanoscale show significant size-dependent behavior [2–4]. Therefore, the surface effect must be considered for the analysis of materials and structures at nanoscale. Some researchers applied atomistic simulation to study the size-dependent properties of nanostructures [5–7]. However, this method is difficult to apply to the analysis of NEMS with complex geometries, due to the limit of the available computational power. It is essential to find an efficient tool to analyze the mechanics behavior of nanoscale structures. Gurtin and Murdoch [8,9] proposed a modified continuum theory which incorporates the surface/interface effects into the traditional continuum mechanics. This theory has been widely used to study the mechanics response of nanoscale structures. For examples, Lim and He [10] proposed a continuum model to analyze the bending behaviors of thin elastic nanoplate of nanoscale thickness. Lu et al. [11] proposed a size-dependent thin plate model by complementing Lim and He's model. Liu and Rajapakse [12] studied the static and dynamic response of nanoscale beams based on the Gurtin–Murdoch theory. Assadi et al. [13] studied the size-dependent dynamic response of nanoplates by using the Gurtin–Murdoch theory. Wang and Feng [14,15] studied the influence of the surface effect on the buckling and vibration behaviors of nanowires. Fang et al. [16] studied the influence of the surface/interface effect on the dynamic stress of two interacting cylindrical nano-inhomogeneities under compressive waves based on the surface/interface elasticity theory.

Analytical solutions are impossible for the structures with complex geometry and boundary conditions. It is necessary to develop a versatile numerical model, such as, the finite element (FE) method and the boundary element (BE) method. Wei et al. [17] proposed a kind of surface element for a two dimensional continuum FE model to take into account the surface elastic effect (based on the Gurtin–Murdoch theory). Tian and Rajapkse [18] studied the mechanics of nanoscale inhomogeneities in an elastic matrix by proposed a FE model. Feng et al. [19] developed a 3D FE model to study the resonant properties of silicon nanowires. Liu et al. [20] proposed a Galerkin-type finite element of the thin and thick beam with the surface effect. In addition, Dong and Pan [21] proposed a BE method to analyze the stress field in nano-inhomogeneities with the surface/interface effect.

Nanoplates with complex geometry, boundary and loading conditions are often used in NEMS. Such complicated structural systems cannot be studied by analytical models. However, an efficient numerical model is not available at this moment for the analysis of nanoscale plates. In the present paper, a finite element model is developed to analyze the bending behavior of nanoplates with consideration of surface residual stress and surface elasticity. The present FE model is based on the plate mathematical model developed by Lu et al. [11] by using the Gurtin–Murdoch surface

* Corresponding author. Tel.: +86 755 26033490.
*E-mail addresses:* wangbl2001@hotmail.com, wangbl@hitsz.edu.cn (B.L. Wang).

0168-874X/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.finel.2013.05.007

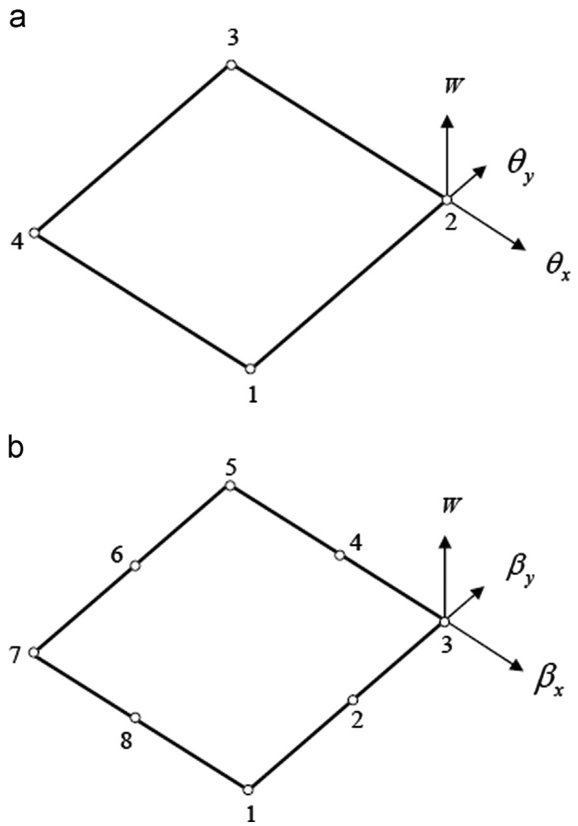


**Fig. 1.** (a) Four-node plate element; (b) eight-node plate element.

elasticity theory. The accuracy and convergence of the present finite element model are verified by comparing the results with the available analytical solutions. The model is used to investigate the influence of residual surface stress and surface elasticity on bending and free vibration of nanoplates with different boundary conditions.

## 2. Finite element formulation

The static equilibrium equations for the bulk of the plate without considering body force are $\sigma_{ij,j}=0$, where $\sigma_{ij}$ denote stresses of the bulk. According to Ref. [8], the surface stresses satisfy the following relations:

$$\tau^{\pm}_{\beta i,\beta}-\sigma^{\pm}_{i3}=0 \tag{1}$$

where $\tau^{\pm}_{\beta i}$ denote the surface stresses on the surface $S^{\pm}$. Using $\sigma_{ij,j}=0$ and Eq. (1) we can obtain the equilibrium equations of plate with the surface effect [11]

$$N^*_{i\beta,\beta}+q=I\ddot{u}_i \tag{2a}$$

$$M^*_{\alpha\beta,\beta}-N_{\alpha 3}=J\ddot{u}_\alpha \tag{2b}$$

where $I=\int_{-h/2}^{h/2}\rho\, dz$, $J=\int_{-h/2}^{h/2}\rho z^2\, dz$, $N^*_{ij}=N_{i\beta}+\tau^+_{\alpha i}+\tau^-_{\alpha i}$, $M^*_{\alpha\beta}=M_{\alpha\beta}+(h/2)(\tau^+_{\beta\alpha}-\tau^-_{\beta\alpha})$, and $N_{ij}=\int_{-h/2}^{h/2}\sigma_{ij}\, dz$ and $M_{ij}=\int_{-h/2}^{h/2}\sigma_{ij}z\, dz$.

According to Refs. [4,8], linear constitutive equations for the surface are

$$\tau_{\alpha\beta}=\tau^0_{\alpha\beta}+C^s_{\alpha\beta\gamma\delta}\varepsilon^s_{\gamma\delta},\quad \tau_{\alpha 3}=\tau^0_\alpha u_{3,\alpha} \tag{3}$$

where$\tau^0_{\alpha\beta}$, $C^s_{\alpha\beta\gamma\delta}$ and $\varepsilon^s_{\gamma\delta}$ are the receptivity, the residual surface stresses, the surface elastic constants and surface strains. Both $\tau^0_{\alpha\beta}$ and $C^s_{\alpha\beta\gamma\delta}$ can be obtained from atomistic calculations.

### 2.1. Static bending of Kirchhoff plate

According to the Kirchhoff plate theory, the displacement components are $u_\alpha=-zu_{3,\alpha}$ and $u_3=w$. Using Eq. (2), we obtained

$$M^*_{x,xx}+2M^*_{xy,xy}+M^*_{y,yy}+2(\tau_{xx}w_{,xx}+\tau_{yy}w_{,yy})+q=0 \tag{4}$$

where $\{M^*_{,xx}\quad M^*_{,yy}\quad M^*_{,xy}\}^T=[\mathbf{D}]\boldsymbol{\psi}+(h^2/2)[\mathbf{C}^s]\boldsymbol{\psi}$, $\boldsymbol{\psi}=\{w_{,xx}\quad w_{,yy}\quad -2w_{,xy}\}^T$, and the material property matrices $[\mathbf{D}]$ and $[\mathbf{C}^s]$ are given in Appendix A. For static bending of the Kirchhoff plate, applying Galerkin's weighted residual method to Eq. (4) gives

$$\iint_A(M^*_{x,xx}+2M^*_{xy,xy}+M^*_{y,yy}+2(\tau_{xx}w_{,xx}+\tau_{yy}w_{,yy})+q)\overline{w}\, dA=0 \tag{5}$$

Using Green's theorem, we get the weak form of Eq. (5) as

$$\begin{aligned}&\iint_A(M^*_x\overline{w}_{,xx}+2M^*_{xy}\overline{w}_{,xy}+M^*_y\overline{w}_{,yy})dA\\&+2\iint_A(\tau_{xx}w_{,xx}+\tau_{yy}w_{,yy})\overline{w}\, dA+\iint_A q\overline{w}\, dA\\&+\int_S(V^*_xn_x+V^*_yn_y)\overline{w}\, dS-\int_S(M^*_xn_x+M^*_{xy}n_y)\overline{w}_{,x}\, dS-\int_S(M^*_{xy}n_x\\&+M^*_yn_y)\overline{w}_{,y}\, dS=0\end{aligned} \tag{6}$$

where $V^*_x=M^*_{x,x}+M^*_{xy,y}$ and $V^*_y=M^*_{xy,x}+M^*_{y,y}$.

The boundary conditions are usually expressed in terms of directions that are normal and tangent to the boundaries. These are the derivatives in the normal direction $\partial\overline{w}/\partial n$ and in the tangential direction $\partial\overline{w}/\partial T$. Here $n$ is the outward unit vector normal to the boundary of the plate, whose components are $n_x$ and $n_y$, $T$ is the unit vector tangent to the boundary of the plate, whose $x$ and $y$ components are $-n_y$ and $n_x$. By these definitions, $\overline{w}_n=n_x\overline{w}_x+n_y\overline{w}_y$, $\overline{w}_T=-n_y\overline{w}_x+n_x\overline{w}_y$ and $n_x^2+n_y^2=1$. The last two boundary integrals in Eq. (6) can now be written as

$$\begin{aligned}&\int_S[(M^*_xn_x+M^*_{xy}n_y)(n_x\overline{w}_n-n_y\overline{w}_T)+(M^*_{xy}n_x+M^*_yn_y)(n_y\overline{w}_{,n}+n_x\overline{w}_{,T})]dS\\&=\int_S[(M^*_xn_x^2+M^*_yn_y^2+2M^*_{xy}n_xn_y)\overline{w}_n\\&\quad+(-M^*_xn_xn_y+M^*_yn_xn_y+M^*_{xy}(n_x^2-n_y^2))\overline{w}_T]dS\\&=\int_S M^*_n\overline{w}_n\, dS+\int_S M^*_T\overline{w}_T\, dS\end{aligned} \tag{7}$$

Finally, the weak form Eq. (6) can be rewritten as

$$\begin{aligned}&\iint_A\overline{\boldsymbol{\psi}}^{\mathrm{T}}[\mathbf{D}]\boldsymbol{\psi}dA+\frac{h^2}{2}\iint_A\overline{\boldsymbol{\psi}}^{\mathrm{T}}[\mathbf{C}^s]\boldsymbol{\psi}dA-2\iint_A(\tau_{xx}w_{xx}+\tau_{yy}w_{yy})\overline{w}\, dA\\&-\iint_A q\overline{w}\, dA-\int_s(V^*_n+M^*_{T,T})\overline{w}\, ds+\int_s M^*_n\overline{w}_n\, ds=0\end{aligned} \tag{8}$$

where

$$\overline{\boldsymbol{\psi}}=\begin{bmatrix}\overline{w}_{,xx} & \overline{w}_{,yy} & 2\overline{w}_{,xy}\end{bmatrix}^{\mathrm{T}} \tag{9}$$

Consider a four-node finite element with three nodal degrees of freedom per node, i.e.,$w$, $\theta_x$ and $\theta_y$ as shown in Fig. 1(a).

The element nodal displacement vector is

$$\mathbf{u}_e=\begin{bmatrix}w_1 & \theta_{x1} & \theta_{y1} & \dots & w_4 & \theta_{x4} & \theta_{y4}\end{bmatrix}^T \tag{10}$$

The displacement vector of the element and the vector of element curvatures are, receptivity, $w=\mathbf{N}^T\mathbf{u}_e$ and $\boldsymbol{\psi}=[\mathbf{B}]^{\mathrm{T}}\mathbf{u}_e$. Here the shape function $\mathbf{N}$ and the geometry matrix $[\mathbf{B}]$ are given in Appendix A. Substituting Eq. (9) and the weighting functions ($\overline{w}\to\mathbf{N}_i$ and $\overline{\boldsymbol{\psi}}\to[\mathbf{B}]^{\mathrm{T}}$) into Eq. (8), we obtain

$$[\mathbf{k}_e]=[\mathbf{k}_b]+[\mathbf{k}_s]+[\mathbf{k}_\tau] \tag{11}$$

where

$$[\mathbf{k}_b]=\iint_A[\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]dA \tag{12a}$$

$$[\mathbf{k}_s]=\frac{h^2}{2}\iint_A[\mathbf{B}]^T[\mathbf{C}^s][\mathbf{B}]dA \tag{12b}$$