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Sensitivity and Hessian matrix analysis of power spectral density functions for uniformly modulated evolutionary random seismic responses

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ABSTRACT

This paper describes a numerical method for calculation of the sensitivity and Hessian matrix of the response PSD functions of structures subjected to uniformly modulated evolutionary random seismic excitation. The method is formulated based on the pseudo excitation method and Newmark method. The evolutionary non-stationary random response analysis is converted into step-by-step integration computations using the pseudo excitation method. The formulas of the pseudo responses, their first and second derivatives with respect to the structural design variables are derived based on the Newmark method. The PSD functions, their sensitivity and Hessian matrix are calculated using the pseudo responses, their first and second derivatives, respectively. Then the computation procedure of sensitivity and Hessian matrix of PSD functions is given in detail. Finally, the PSD functions' sensitivity and Hessian matrix analysis of a three-story, two-bay planar frame subjected to the uniformly modulated evolutionary random earthquake ground motion has been studied to elucidate the proposed method.

1. Introduction

The sensitivities of structural dynamic responses are the essential information for the gradient-based optimization methods and needed in structural dynamic reliability analysis, optimization and identification, etc. The sensitivity [1–3] and Hessian matrix [4,5] analysis of structures subjected to transient dynamic loads are studied but restricted to deterministic dynamic loads. However, in reality, the major dynamic loads, i.e., earthquake, wind and wave, on structures are stochastic process in nature. The responses of a structure subjected to such uncertain loads are also stochastic in nature [6,7]. The concept of stochastic sensitivity has been early proposed by Socha [8] and Szopa [9]. Several papers have been devoted to the response sensitivity analysis of structures subjected to stochastic processes. As an example, Chaudhuri and Chakraborty [10] dealt with the response sensitivity evaluation in double frequency domain of structures subjected to the non-stationary earthquake motion. Benfratello et al. [11] proposed a procedure, in the time domain, to evaluate the sensitivity of the statistical moments of the structural response for stationary Gaussian and non-Gaussian white input processes. Cacciola et al. [12] presented a method for the evaluation of the response sensitivity of both classically and non-classically damped discrete linear structural systems under stochastic actions. However, there is little work published on Hessian matrix analysis of the responses of structures subjected to stochastic excitation. The sensitivity and Hessian matrix are often simultaneously used in the solution of various problems. In structural optimal design, they are often required to select a search direction and search step in some mathematics programming methods, such as Newton's method and second order optimization methods [13].

Power spectral density functions, i.e., PSD functions, are very important physical quantity for random seismic responses [14,15]. For example, for zero mean stationary random processing, variance can be calculated if the Auto-PSD function is first obtained. Furthermore, if the random seismic response process is normal stochastic process, the probability density function and probability distribution function are completely determined too. Sensitivity analysis of PSD functions for random seismic responses deals with calculation of the first derivatives of PSD functions with respect to the structural design variables and Hessian matrix analysis of PSD functions for random seismic responses deals with calculation of the second derivatives of PSD functions with respect to the structural design variables.

The purpose of this paper is to develop a numerical method for calculation of the sensitivity and Hessian matrix of the response PSD functions of structures subjected to the uniformly modulated evolutionary random excitation. The paper is arranged as follows. In Section 2, the evolutionary non-stationary random response analysis is converted into step-by-step integration computations

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using the pseudo excitation method [16,17]. The formulas for the pseudo responses and PSD function matrix are derived based on Newmark-β method. In Section 3, the formulas for the first derivatives of PSD functions are derived by direct differentiation. In Section 4, the formulas for the second derivatives of PSD functions are also derived by direct differentiation. In Section 5, the procedure of calculating PSD functions, their first and second derivatives is given in detail. Finally, the PSD functions' sensitivity and Hessian matrix analysis of a three-story, two-bay planar frame subjected to uniformly modulated evolutionary random earthquake ground motion is demonstrated with the proposed method.

2. Formulas for PSD functions

When a structure is subjected to uniformly modulated evolutionary random ground acceleration excitation $\ddot{x}_g(t)g(t)$, the governing equation of structural dynamics can be expressed as follows:

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(t)+\boldsymbol{C}\dot{\boldsymbol{x}}(t)+\boldsymbol{K}\boldsymbol{x}(t)=-\boldsymbol{M}\boldsymbol{E}_u\ddot{x}_g(t)g(t) \tag{1}$$

With initial conditions

$$\begin{cases}\boldsymbol{x}(0)=0\\ \dot{\boldsymbol{x}}(0)=0\end{cases} \tag{2}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ are the structural mass matrix, damping matrix and stiffness matrix, respectively. $\boldsymbol{x}(t)$, $\dot{\boldsymbol{x}}(t)$ and $\ddot{\boldsymbol{x}}(t)$ are the displacement, velocity and acceleration vectors relative to the ground, respectively. $\boldsymbol{E}_u$ is a vector. For planar frame structures, $\boldsymbol{E}_u=[1\ 0\ 0\ 1\ 0\ 0\ \cdots\ 1\ 0\ 0]^T$. $\ddot{x}_g(t)$ is the ground horizontal acceleration time history and $g(t)$ is the modulated function

$$g(t)=A[e^{-\gamma_1 t}-e^{-\gamma_2 t}] \tag{3}$$

where A, γ_1 and γ_2 are constants.

Rayleigh damping is used in this work, the structural damping matrix is

$$\boldsymbol{C}=\alpha_1\boldsymbol{M}+\alpha_2\boldsymbol{K} \tag{4}$$

where

$$\alpha_1=\frac{2\omega_1\omega_2(\zeta_1\omega_2-\omega_1\zeta_2)}{\omega_2^2-\omega_1^2} \tag{5}$$

$$\alpha_2=\frac{2(\zeta_2\omega_2-\zeta_1\omega_1)}{\omega_2^2-\omega_1^2} \tag{6}$$

where ω_1 and ω_2 are the first and second natural frequency of the structure, respectively. ζ_1 and ζ_2 are the first and second mode damping ratios, respectively.

Pseudo excitation method converts any stationary random response analysis into harmonic response analyses, and converts any evolutionary non-stationary random response analysis into step-by-step integration computations. The pseudo excitation of ground acceleration is constructed as follows:

$$\tilde{f}(\omega,t)=\sqrt{S_{\ddot{x}_g}(\omega)}g(t)e^{i\omega t} \tag{7}$$

where $S_{\ddot{x}_g}(\omega)$ is the power spectrum of ground acceleration, it can be presented by the Kanai–Taijimi expression [18]

$$S_{\ddot{x}_g}(\omega)=\frac{1+4\varsigma_g^2(\omega/\omega_g)^2}{[1-(\omega/\omega_g)^2]^2+4\varsigma_g^2(\omega/\omega_g)^2}S_0 \tag{8}$$

where ω_g is the natural ground frequency, ς_g is the viscous damping for the ground and S_0 is the intensity parameter.

Eqs. (1) and (2) become

$$\boldsymbol{M}\ddot{\tilde{\boldsymbol{x}}}(\omega,t)+\boldsymbol{C}\dot{\tilde{\boldsymbol{x}}}(\omega,t)+\boldsymbol{K}\tilde{\boldsymbol{x}}(\omega,t)=-\boldsymbol{M}\boldsymbol{E}_u\tilde{f}(\omega,t) \tag{9}$$

With initial conditions

$$\begin{cases}\tilde{\boldsymbol{x}}(\omega,0)=0\\ \dot{\tilde{\boldsymbol{x}}}(\omega,0)=0\end{cases} \tag{10}$$

Eqs. (9) and (10) must be satisfied for all time period $t\in[0,T_e]$, T_e is the duration of earthquake, at any frequency ω. The most widely used family of direct time integration methods for solving Eq. (9) is the Newmark family of methods. The Newmark method can be formulated by considering equilibrium at any discrete time $t+\Delta t$, and is given by the following equation:

$$\boldsymbol{M}\ddot{\tilde{\boldsymbol{x}}}(\omega,t+\Delta t)+\boldsymbol{C}\dot{\tilde{\boldsymbol{x}}}(\omega,t+\Delta t)+\boldsymbol{K}\tilde{\boldsymbol{x}}(\omega,t+\Delta t)=-\boldsymbol{M}\boldsymbol{E}_u\tilde{f}(\omega,t+\Delta t) \tag{11}$$

The Newmark-β method is an implicit technique, which consists of the following finite difference assumptions with regard to the evolution of the approximate solution:

$$\tilde{\boldsymbol{x}}(\omega,t+\Delta t)=\tilde{\boldsymbol{x}}(\omega,t)+\Delta t\dot{\tilde{\boldsymbol{x}}}(\omega,t)+\Delta t^2[((1/2)-\beta)\ddot{\tilde{\boldsymbol{x}}}(\omega,t)+\beta\ddot{\tilde{\boldsymbol{x}}}(\omega,t+\Delta t)] \tag{12}$$

$$\dot{\tilde{\boldsymbol{x}}}(\omega,t+\Delta t)=\dot{\tilde{\boldsymbol{x}}}(\omega,t)+\Delta t[(1-\delta)\ddot{\tilde{\boldsymbol{x}}}(\omega,t)+\delta\ddot{\tilde{\boldsymbol{x}}}(\omega,t+\Delta t)] \tag{13}$$

where any particular choice of the parameters β and δ determines the stability and accuracy characteristics of the solution. In this work, we chose the parameters $\delta\geq 0.5$ and $\beta=0.25(0.5+\delta)^2$. We also define the integral constants: $a_0=1/(\beta\Delta t^2)$, $a_1=\delta/(\beta\Delta t)$, $a_2=1/(\beta\Delta t)$, $a_3=(1/(2\beta))-1$, $a_4=(\delta/\beta)-1$, $a_5=(\Delta t/2)((\delta/\beta)-2)$, $a_6=\Delta t(1-\delta)$, $a_7=\delta\Delta t$. The parameters β and δ will be replaced by those constants in the following formulas.

In addition to Eqs. (12) and (13) the equilibrium equation (11) at time station $t+\Delta t$ is considered. This way a system of equations is formed for the determination of the three unknowns $\tilde{\boldsymbol{x}}(\omega,t+\Delta t)$, $\dot{\tilde{\boldsymbol{x}}}(\omega,t+\Delta t)$ and $\ddot{\tilde{\boldsymbol{x}}}(\omega,t+\Delta t)$, assuming that the pseudo displacement, velocity and acceleration vectors at the previous time station t have already been computed. Thus, the solution for the pseudo displacement vector is

$$\boldsymbol{K}^*\tilde{\boldsymbol{x}}(\omega,t+\Delta t)=\tilde{\boldsymbol{F}}^*(\omega,t+\Delta t) \tag{14}$$

where

$$\boldsymbol{K}^*=\boldsymbol{K}+a_0\boldsymbol{M}+a_1\boldsymbol{C} \tag{15}$$

and

$$\begin{aligned}\tilde{\boldsymbol{F}}^*(\omega,t+\Delta t)=&-\boldsymbol{M}\boldsymbol{E}_u\tilde{f}(\omega,t+\Delta t)+\boldsymbol{M}[a_0\tilde{\boldsymbol{x}}(\omega,t)+a_2\dot{\tilde{\boldsymbol{x}}}(\omega,t)+a_3\ddot{\tilde{\boldsymbol{x}}}(\omega,t)]\\ &+\boldsymbol{C}[a_1\tilde{\boldsymbol{x}}(\omega,t)+a_4\dot{\tilde{\boldsymbol{x}}}(\omega,t)+a_5\ddot{\tilde{\boldsymbol{x}}}(\omega,t)]\end{aligned} \tag{16}$$

The matrix $\boldsymbol{K}^*$ is positive definite and symmetric. So it can be uniquely done via triangular factorization

$$\boldsymbol{K}^*=\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T \tag{17}$$

where $\boldsymbol{L}$ is lower triangular matrix and $\boldsymbol{D}$ is diagonal matrix.

Therefore, Eq. (14) can be solved as follows:

$$\tilde{\boldsymbol{x}}(\omega,t+\Delta t)=(\boldsymbol{L}^{-1})^T\boldsymbol{D}^{-1}\boldsymbol{L}^{-1}\tilde{\boldsymbol{F}}^*(\omega,t+\Delta t) \tag{18}$$

The pseudo accelerations, $\ddot{\tilde{\boldsymbol{x}}}(\omega,t+\Delta t)$, which are required for the computations at the next time station, can be calculated as follows:

$$\ddot{\tilde{\boldsymbol{x}}}(\omega,t+\Delta t)=a_0[\tilde{\boldsymbol{x}}(\omega,t+\Delta t)-\tilde{\boldsymbol{x}}(\omega,t)]-a_2\dot{\tilde{\boldsymbol{x}}}(\omega,t)-a_3\ddot{\tilde{\boldsymbol{x}}}(\omega,t) \tag{19}$$

while the pseudo velocities, $\dot{\tilde{\boldsymbol{x}}}(\omega,t+\Delta t)$, can be obtained directly from Eq. (13)

$$\dot{\tilde{\boldsymbol{x}}}(\omega,t+\Delta t)=\dot{\tilde{\boldsymbol{x}}}(\omega,t)+a_6\ddot{\tilde{\boldsymbol{x}}}(\omega,t)+a_7\ddot{\tilde{\boldsymbol{x}}}(\omega,t+\Delta t) \tag{20}$$

The PSD function matrix is

$$\boldsymbol{S}_x(\omega,t+\Delta t)=\tilde{\boldsymbol{x}}^*(\omega,t+\Delta t)\cdot[\tilde{\boldsymbol{x}}(\omega,t+\Delta t)]^T \tag{21}$$

where '*' is a complex conjugate.