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Development of a new finite element for plate and shell analysis by application of generalized approach to patch test

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ABSTRACT

A triangular flat finite element with three nodes in the corners has been developed for analysis of Kirchhoff plates and shells within the framework of the extended interpretation of the Irons patch test. The condition of the bending displacements continuity in the case of non-constant strains is supposed to be violated not only on the boundaries but inside the area of an element as well. In general the typical approximations of the finite element method are implemented here in the limit of the infinitesimal elements. Thus it is possible to obtain the relatively simple but quite effective finite elements. The numerical results based on the classical tests demonstrate the high accuracy and quick convergence of the solutions obtained with the help of the presented finite element.

1. Introduction

The convergence of the finite element method is provided under minimization of the Lagrangian functional even if the conditions of the displacement continuity on the element boundaries are violated when the applied approximations meet the requirements of the Irons patch test [1–3]. The Irons approach allows to satisfy the classical conditions of the rigid body motion and the constant strain for an individual finite element which are necessary for the convergence of the given method although it relaxes the requirement concerning the continuity of the finite element system. If it is possible to reproduce the constant strain by the finite element system then the obtained results will converge to exact solution with the mesh refinement. This idea has allowed to prove the convergence of a number of the finite elements for the bending analysis of thin plates and to justify the possible application of the flat elements for shell analysis.

At the same time the development of the finite elements intended for the analysis of thin plates and shells remains quite difficult even in view of the relaxed requirements of the patch test. In particular it is necessary for the decision of this problem to employ the subelements, to specify the nodes with a different number of degrees of freedom, to consider the second derivatives of displacements in the nodes [1].

The application of mixed formulations of the finite element method is a major direction for deriving the effective finite elements for plate bending analysis [1,2,4]. Thus Pian and Tong [5] proposed the hybrid stress method that includes the Lagrange multipliers in the modified principle of complementary potential energy to force the interelement equilibrium. Then Tong [6] developed the displacement hybrid method based on the modified principle of minimum potential energy. Anderheggen [7] considered the analogous approach to avoid normal slope discontinuities along the sides of the elements. The plate displacements are subjected to slope continuity conditions acting by introducing Lagrange multipliers to the minimum potential energy problem. In the discrete Kirchhoff family elements [8–14], as distinct from the classical finite element formulation, the assumptions of the Kirchhoff plate theory are constrained only at some discrete points. In addition, here the individual approximations on bending and angles of rotation are considered. These approximations are linked by a collocation method and the problem of minimization of the mixed functional using Lagrangian multipliers or penalty procedures is solved. The collocation method has also demonstrated a rather high effectiveness for solution of plate bending problems in combination with the Galerkin finite element technique [15]. The MITC plate element family [4,16–19] dealt with the Reissner–Midling theory is based on the mixed discrete variation principle. The functional of this principle includes a reduction operator relaxing the constraint of vanishing shear strain in the discrete displacements. The deflection and angles of rotation are approximated separately as in the discrete Kirchhoff approach. Some components of the reduced shear strain are taken equal in several points to the same components of the shear strain calculated from the displacements.

The relatively simple triangular finite elements with three corner nodes for thin plate analysis where three degrees of freedom in each node are considered represent the great interest

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for practice. At present there exist a number of elements of the given kind. A non-conforming finite element developed by an incomplete polynomial of the third degree is widely known [20]. This element does not pass the patch test for arbitrary mesh configuration. Nevertheless, it is popular in practical application [1]. The given finite element has been modified [21] by introduction of the additional boundary integral into the virtual work statement. It began to satisfy the patch test as a result. A number of other modifications have been considered [1]. Thus Specht [22] achieved the satisfaction of the patch test by introduction of the fourth-order terms into the shape functions.

Stricklin et al. [9] and Dhatt [10] obtained such triangular element with the help of the discrete Kirchhoff constraints. Batoz et al. [23] considered the following effective triangular plate bending elements with nine degrees of freedom in three corner nodes: the DKT element based on discrete Kirchoff theory assumptions, the HSM element based on the hybrid stress method, and the SRI element based on selective reduced integration scheme that includes transverse shear deformation. They note that the DKT element gives better results for Kirchoff plates than the HSM and SRI elements. The MITC 3-node element for the plate bending analysis with the same node variables was also constructed but it locks for a specific mesh pattern [24].

Long [25] developed three conforming triangular finite elements for thin plates with nine degrees of freedom with the help of the modified principle of potential energy by using compatibility conditions for nodal deflection at each node and generalized compatibility conditions for average deflection and average normal slope along each side of the element.

Shell finite elements can be grouped into following kinds: flat element formed with combination of the shape functions describing the bending displacements and displacements in the plane of the element (facet element), continuum based shell element, solid 3D element and a 2D element based on a shell theory [1,2,4]. Exactly the facet elements capable to be built into the automated design easily have found the most frequent application in the packages developed to solve a great variety of problems concerning the investigation of the thin wall structure deformations. Nevertheless, the available facet elements fail to satisfy the convergence properties for some difficult tests, in particular for the hyperbolic parabolic [4]. At the same time the well-known shell elements using a general shell theory or general 3D continuum mechanics approach included shell theory assumptions demonstrate the good convergence for the standard bending dominated and membrane dominated tests [1,4,26–30].

In the present paper, the problem of the development of the effective triangular facet element with three corner nodes for plate and shell analysis is being solved on the basis of the modified approach to the finite element method in the framework of the application of the Lagrangian variational principle. Unlike the traditional scheme of the creation of the finite elements, the suggested technique requires the satisfaction of the conditions of continuity in all points of the element only for constant strain. Hereby there is no need to describe explicitly the displacement functions in the whole area of the finite element that does not violate the approach to the finite element method as to one of the possible forms of the variational-difference procedure [2,31]. The approximation of displacements is implemented on individual linear segments which can be located on the boundary or inside the area of the element. Nevertheless, under the above-mentioned approach to the description of the unknown functions the requirement to satisfy the conditions of the patch test is kept. The procedure will be called a limit scheme of the finite element method as the typical approximations of the latter are generally implemented here only for the infinitesimal elements, where the length of the longest side of a triangle can be considered as the determining geometrical parameter of the triangular element [3]. The limit scheme is introduced here for the plate bending descriptions while the membrane displacements are approximated with the help of the well-known approaches. This element demonstrates good convergence for plate and shell problems.

2. Description of bending deformations of plates

A triangular Kirchhoff plate bending element has been developed on the basis of the limit scheme of the finite element method. The finite element shown in Fig. 1 has nodes 1, 2 and 3 in the corner points where the deflections and angles of rotation around axes O_{x_1} and O_{x_2} of the Cartesian coordinate system $O_{x_1x_2x_3}$ are generally considered. Extra points 4, 5 and 6 have to be introduced in the centres of the triangle sides.

The generalized strains χ_1, χ_2, χ_3 of the plate ($\chi_1 = -\partial^2\delta_{x_3}/\partial x_1^2$, $\chi_2 = -\partial^2\delta_{x_3}/\partial x_2^2$, $\chi_{12} = -2\partial^2\delta_{x_3}/\partial x_1\partial x_2$) [1] in point 4 can be expressed through the generalized nodal displacements of the finite element nodes, where δ_{x_3} is O_{x_3} projection of the deflection vector. For this purpose the Cartesian coordinate systems $\tilde{O}_{y_1y_2y_3}$ and $\tilde{O}''_{z_1z_2z_3}$ are assumed as shown in Fig. 1. At first we approximate the deflection and the angle of rotation around axis $\tilde{O}_{y_1}$ on segment 1–2. Then the deflection on segment 3–4 is approximated with the help of the obtained results. This approach allows to provide the equality of the deflection and the angles of rotation in point 4 for adjacent finite elements *A* and *B* (see Fig. 1). On the basis of the approximations on segment 1–2 the values of the second derivatives $\partial^2\delta_{y_3}/\partial y_1^2$ and $\partial^2\delta_{y_3}/\partial y_1 y_2$ in this point are expressed through the generalized displacements of nodes 1 and 2, where δ_{y_3} is $\tilde{O}_{y_3}$ projection of the deflection vector. With the help of the approximation on segment 3–4 the value of function $\partial^2\delta_{y_3}/\partial z_1^2$ in point 4 is expressed through the generalized displacements of nodes 1, 2 and 3. Following this the transfer to the generalized strains χ_1, χ_2 and χ_3 in the coordinate system $O_{x_1x_2x_3}$ is implemented.

Consider that the deflection varies on segments 1–2 and 3–4 by the cube laws and the angle of rotation around axis $\tilde{O}_{y_1}$ varies on segment 1–2 by the linear law. We can write [32]

$$\delta_{y_3}^{(12)} = \left(1-\frac{3y_1^2}{d_{12}^2}+\frac{2y_1^3}{d_{12}^3}\right)\delta_{y_3}^{(1)} - y_1\left(1-\frac{y_1}{d_{12}}\right)^2\theta_{y_2}^{(1)} + \frac{y_1^2}{d_{12}^2}\left(3-\frac{2y_1}{d_{12}}\right)\delta_{y_3}^{(2)} + y_1\left(1-\frac{y_1}{d_{12}}\right)\frac{y_1}{d_{12}}\theta_{y_2}^{(2)}, \quad (1)$$

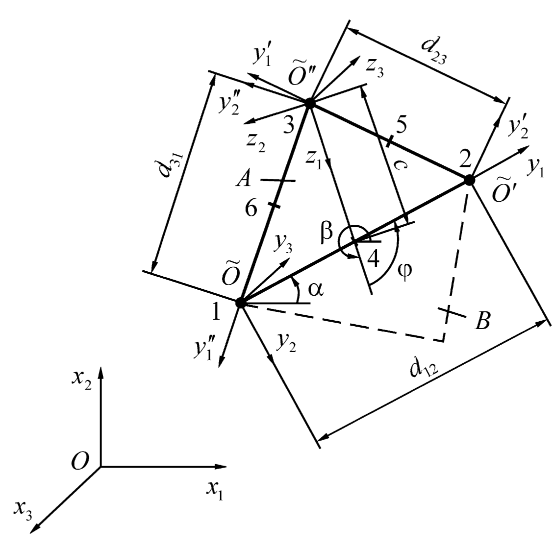

Fig. 1. Triangular three-node finite element *A* abutting on the finite element *B*.