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Hybrid mesh smoothing based on Riemannian metric non-conformity minimization

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ABSTRACT

A smoothing method specifically designed to treat hybrid meshes is presented in this paper. This method, based on Riemannian metric comparison, minimizes a cost function constructed from a measure of metric non-conformity that compares two metrics: the metric that transforms a given element into its reference element and a specified Riemannian metric that contains the desired target size and shape of each element. This combination of metrics allows the proposed mesh smoothing method to be cast in a very general frame, valid for any dimension and type of element. Numerical examples show that the proposed method generates high quality meshes as measured both in terms of element characteristics and in terms of orthogonality at the boundary and overall smoothness, when compared to other known methods.

1. Introduction

In the context of numerical simulations, particularly in computational fluid dynamics (CFD), the concept of mesh quality is always an issue. Smoothing is a mesh modification method that can be used to increase mesh quality in many ways. Most often, simple smoothing algorithms are used after initial mesh generation or topological modifications to an existing mesh, in order to equidistribute variations of size or shape globally or locally, see [1,2] for examples.

In this paper, a mesh smoothing method driven by the minimization of metric non-conformity is proposed. The presented method, instead of optimizing size or shape functions, directly compares an element's current metric to a desired target metric. These metrics contain, in a single matrix entity, details on local size and shape. Since the algorithm is only dependant on a specified metric, it can be used in different settings such as initial mesh generation, where the specified metric is constructed from geometric information, or in *a posteriori* adaptation, where the metric is computed from a numerical solution. Assuming that a correctly defined metric is specified, this paper explains how a mesh smoothing method can be devised to generate high quality meshes with respect to the desired metric, while respecting many constraints for the mesh such as constant number of vertices and constant connectivity between vertices of the mesh. The main contribution of this paper is to propose a method that is applicable to hybrid meshes, which comprise elements of different types, and to show how minimization of metric non-conformity can be incorporated into a convergent mesh adaptation algorithm.

The first section of this paper presents some of the works related to mesh smoothing and discusses why a new smoothing algorithm is needed for industrial applications, which simultaneously accounts for both size and shape of the elements. The concepts of Riemannian metrics and non-conformity are explained next, in Section 3. This paper then goes on to explain the smoothing method used to optimize the non-conformity of a mesh (Section 4) and presents an algorithm based on the prototype presented in [3]. Numerical examples that illustrate the versatility of the method and the quality of the resulting meshes are presented in the final part of this paper for different types of meshes, and conclusions are drawn.

2. Mesh optimization by smoothing

Most smoothing methods can be separated into two categories: methods that optimize size distribution and methods that optimize element shape.

In the size distribution methods category, the most common type of smoothing is certainly Laplacian smoothing, where a vertex is moved to the center of its neighbors. Examples of other size distribution methods include physical analogies such as the spring analogy [4,5] and particle potential minimization [6], methods based on the elliptical Poisson system [7–10] as well as "center of mass" methods [11,12]. These methods have been used in adaptive environments using weight functions or concentration functions that allow for

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spacing or size specification. Their main drawback is that they provide very little control over element shape, since they are only based on the measure of distance between points. They are usually not appropriate when orthogonality or other shape properties are desired. Moreover, these methods are subject to geometric tensions. This means that vertices are attracted in concave corners of the geometry, and this pulling effect can even result in the mesh folding outside the geometry, since optimal positioning of nodes is based on length and is not, by definition, aware of domain boundaries. This behavior can be controlled using constraints on the optimization process to enforce boundaries, or concentration functions to reduce tensions. But these processes must often be adjusted somewhat manually for a given class of geometries. This implies that, as they are formulated, these optimization processes do not entirely incorporate the underlying engineering and computational objectives of smoothing.

The second category of methods is shape-based optimization. Some of the best known methods in this category apply complex optimization algorithms to reduce a cost function based, for example, on angular criteria [13,14] or on shape distortion measures such as those presented in [15–17]. Most often, these smoothing methods are used as a final step during mesh generation, to regulate shape variations from an ideal shape, for example, a square for a 2D quadrilateral element. The resulting meshes exhibit very smooth shape distribution. The inherent limitation of these methods lies in the fact that the definition of a perfect element shape is global. When a vertex is moved, the optimization process tries to satisfy a specific shape which is the same over the whole domain and is usually isotropic. These approaches are excellent to correct unsatisfactory shape distortions in a generated mesh. Local specification of shape can be implemented in these methods in order to also adapt the vertex distribution to complex flow characteristics, which locally exhibit highly anisotropic features.

In order to generate a better result, two or more smoothing methods can be combined either by successively applying each one, sequentially or iteratively, or by minimizing a single cost function obtained as an arbitrary combination of several simpler functions. However, this type of combined method results in heuristic approaches that are application and case dependant and thus not as general as desired. In the present work, a single cost function is used, rather than an arbitrary combination of functions, in order to prevent spurious properties in resulting meshes and to eliminate the need for case specific modifications to the function.

From the previous analysis, it becomes apparent that current methods lack one of the two kinds of control, either on size or shape, or combine them in a heuristic manner that does not give intrinsic control on both properties at the same time. In the present work, the goal is to unify these controls into a single target specification, and devise a vertex relocation method capable of satisfying at best this specification. For example, it could be necessary, in the same mesh, to specify highly anisotropic elements to resolve a shock near an airfoil while also specifying, in another region, highly isotropic elements with great size variations to resolve flow vortices at a trailing edge. In this case, a variation of a shape-based approach might seem best suited for the shock region, while a size-based approach would probably yield the best results for the trailing edge region. To obtain high quality meshes with local control of mesh characteristics, a number of desired properties of the smoothing method have been identified. The smoothing method should allow the

1. simultaneous optimization of both element size and shape;
2. specification of variable size and shape targets over the domain;
3. minimization of a single cost function;
4. smoothing of both structured and unstructured meshes possibly containing mixed element types;
5. construction of non-folding meshes without constraining the optimization process;
6. ensuring of the continuity between regions of different cell types (hybrid meshes).

The first three properties can be met using a cost function based on a Riemannian metric, as the next sections will show. Also, since a metric-based specification of the target mesh characteristics is independent of element type, the use of a cost function based on metric comparison ensures that the optimization process is independent of the mesh and element types as well.

Furthermore, the present work aims at developing a general mesh smoothing method that naturally converges toward non-folded meshes. Hence a formulation of the smoothing problem is chosen that lends itself to unconstrained optimization. It is postulated that for the optimization process to naturally result in high quality meshes without constraints, essentially entails that the overall process be specifying a correct form of the mesh smoothing problem. Here, a correct form of the smoothing problem refers to a formulation where element size, element shape, presence of domain boundaries and fixed mesh connectivity are accounted for intrinsically.

3. The concepts of metric and non-conformity

The use of a Riemannian metric as a size and shape specification map for the adaptation of a mesh is a central concept to this paper. It has been first introduced in [18] as a way to describe the size, stretching and orientation of the mesh elements in a single matrix entity. It has been shown in works such as [19,20] that the Riemannian metric allows the control of mesh characteristics through the specification of a single tensor defined on the domain.

A specified metric $\mathscr{M}_S$ can be constructed from *a posteriori* error estimation or user defined functions as well as geometric properties. The Riemannian metric is an entity that can be used in any adaptation process, independent of how it is constructed and what characteristics the user wants to achieve through the adaptation process. The metric field $\mathscr{M}_S$ uses the Hessian of a scalar variable, in the present case velocity magnitude or Mach number, to adapt the mesh to a solution obtained on an initial mesh (see e.g. [21]). The Hessian itself is reconstructed using the quadratic method described and analyzed in [22].

Smoothing using a specified metric involves moving mesh vertices so that each element is as close as possible to the ideal size and shape, as measured in the space defined by the specified metric. These ideal elements are the unit side equilateral triangles or the unit squares in two dimensions and their equivalents in three dimensions. Being of the ideal size and shape in the metric will result in an element being of the specified size, stretching and orientation, according to the metric.

The quality of a mesh can be measured using the non-conformity coefficient presented for simplices in [23,24] and extended to non-simplices in [25]. The central idea is that each element K in the mesh possesses an actual metric. This metric, denoted as $\mathscr{M}_K$, defines the transformation between the element in its present state and a reference element. The reference element, as described in the previous paragraph, is the same element type as element K with unit length edges. Evaluation of $\mathscr{M}_K$ is done using finite element transformations, as described in [25].

The quality of an element is then defined as being optimal when that actual metric is equal to the specified metric:

$$\mathscr{M}_K = \mathscr{M}_S. \tag{1}$$