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Size gradation control of anisotropic meshes

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ABSTRACT

This paper gives a status on anisotropic mesh gradation. We present two 3D anisotropic formulations of mesh gradation. The metric at each point defines a well-graded smooth continuous metric field over the domain. The mesh gradation then consists in taking into account at each point the strongest size constraint given by all these continuous metric fields. This is achieved by a metric intersection procedure. We apply it to several examples involving highly anisotropic meshes.

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1. Introduction

In many engineering applications, it is desirable to generate anisotropic meshes presenting highly stretched elements in adequate directions. Numerous papers have been published on mesh adaptation for numerical simulations in computational solid or fluid mechanics. Among these papers, some have addressed the problem of creating 3D unstructured anisotropic meshes [5,6,10,12,15,17,20,21,23]. In these approaches, different mesh generation methods are considered. Nonetheless all of them are based on the notion of *unit mesh* in a Riemannian metric space.

In the context of numerical simulations based on finite element or finite volume methods, these works have proved the efficiency of unstructured mesh adaptation to improve the accuracy of the numerical solution as well as for capturing the behavior of physical phenomena. In principle, this technique enables to substantially reduce the number of degrees of freedom, thus impacting favorably (i) the cpu time, (ii) the data storing and (iii) the solution analysis (visualization). Moreover, it has been recently proved in [20] that mesh adaptation impacts positively the order of convergence of numerical scheme by computing the numerical solution with a coherent accuracy in the entire domain.

The size prescription in the generation of anisotropic meshes is achieved thanks to metric fields. However, such metric fields may have huge variations making the generation of a unit mesh difficult or impossible, thus leading to poor quality anisotropic meshes. Generating high-quality anisotropic meshes requires to smooth the metric field by bounding its variations in all directions. To this end, a mesh gradation control procedure was introduced in [4]. It consists in reducing the size prescribed at mesh vertices by checking the metric variations along the mesh edges. The authors first describe an isotropic formulation from which they deduce an extension to anisotropic meshes. They present an homothetic reduction dedicated for surface mesh generation and a non-homothetic reduction. In the context of volume meshing, the homothetic leads to inconsistency in the metric reduction during the size gradation procedure. An anisotropic mesh gradation has also been presented in [16]. This procedure considers spectral decomposition and associate eigenvectors together with ad hoc choices. We prefer a formulation that uses directly well-posed operations on metrics. In [22], an isotropic size gradation control has been applied to the generation of multi-patch parametric surface meshes.

In this paper, we give a status on anisotropic mesh gradation. We present two 3D anisotropic formulations of mesh gradation extending the formulation given in [4] and we apply it to several examples involving highly anisotropic meshes. We formulate the problem mathematically by employing the continuous modeling of a mesh proposed in [19]. The metric at each point defines a well-graded smooth continuous metric field over the domain. The mesh gradation then consists in taking into account at each point the strongest size constraint given by all these continuous metric fields. Numerically, in the context of a mesh with a metric field given at its vertices, the idea consists in imposing at each vertex a size constraint related to all the other vertices of the mesh. To this end, all vertices of the mesh span metric fields in the whole domain by growing their metrics at a rate given by the desired gradation coefficient. Then, the reduced metric at a vertex is the intersection between its metric and all these metrics. Unfortunately, this mesh gradation algorithm is intrinsically of quadratic complexity. We thus establish an approximation to solve it in a linear time.

This paper is outlined as follows. In Section 2 the notion of a metric and the metric-based method to generate anisotropic meshes are described. In Section 3 the isotropic mesh gradation is recalled.

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Then, two anisotropic formulations of the mesh gradation are presented, Section 4. In the numerical examples, Section 5, we illustrate the efficiency of the presented method for anisotropic mesh adaptation. We also exemplify that anisotropic mesh gradation can be used for other applications such as the generation of well-graded meshes. Finally, the limits of the proposed approaches are exemplified on an analytical example in Section 6. New mesh gradation procedures improving the previous ones are then proposed to remedy these problems.

2. Metric and mesh generation

In this section, we recall a metric-based method to generate anisotropic meshes. It is based on the notion of Riemannian metric space and on the concept of *unit mesh* initially introduced in [14]. Well-posed operations on metrics are introduced and we discuss the numerical computation of the length of a given path in a metric space.

2.1. Metric notion

A *metric tensor* (or simply a *metric*) $\mathcal{M}$ in $\mathbb{R}^n$ is an $n\times n$ symmetric definite positive matrix. $\mathcal{M}$ is always diagonalizable and can be decomposed as $\mathcal{M} = {}^t\mathcal{R}\Lambda\mathcal{R}$, where $\mathcal{R}$ and Λ are the eigenvectors and the eigenvalues matrices of $\mathcal{M}$, respectively. From this definition, it follows up that the *scalar product* of two vectors in $\mathbb{R}^n$ can be defined related to a metric $\mathcal{M}$ as

$$\langle \mathbf{u},\mathbf{v}\rangle_{\mathcal{M}} = \langle \mathbf{u},\mathcal{M}\mathbf{v}\rangle = {}^t\mathbf{u}\mathcal{M}\mathbf{v}\in\mathbb{R},$$

where the natural dot product of $\mathbb{R}^n$ has been denoted by $\langle .,.\rangle$. Under this notion, the *Euclidean norm* of a vector $\mathbf{u}$ in $\mathbb{R}^n$ according to $\mathcal{M}$ is defined as

$$\|\mathbf{u}\|_{\mathcal{M}} = \sqrt{\langle \mathbf{u},\mathbf{u}\rangle_{\mathcal{M}}} = \sqrt{{}^t\mathbf{u}\mathcal{M}\mathbf{u}}$$

that actually measures the length of vector $\mathbf{u}$ with respect to metric $\mathcal{M}$. A metric $\mathcal{M}$ could be geometrically represented by its associated unit ball, an ellipsoid, defined by

$$\mathcal{E}_{\mathcal{M}} = \left\{\mathbf{p}\;\middle|\;\sqrt{{}^t\mathbf{op}\mathcal{M}\mathbf{op}}=1\right\},$$

where $\mathbf{o}$ is the center of the ellipsoid, see Fig. 1. The main axes are given by the eigenvectors of matrix $\mathcal{M}$ and the radius along each axis is given by the inverse of the square root of the associated eigenvalue.

Definition 2.1. An *Euclidean metric space* is a vector space supplied with a scalar product $\langle .,.\rangle_{\mathcal{M}}$ defined by metric tensor $\mathcal{M}$. We denote it $(\mathbb{R}^n,\mathcal{M})$. The *distance* between two points $\mathbf{p}$ and $\mathbf{q}$ is given by $d_{\mathcal{M}}(\mathbf{p},\mathbf{q}) = \sqrt{{}^t\mathbf{pq}\mathcal{M}\mathbf{pq}}$. Finally, the *length* of a segment $\mathbf{pq}$ is the distance between its extremities: $\ell_{\mathcal{M}}(\mathbf{pq}) = d_{\mathcal{M}}(\mathbf{p},\mathbf{q})$.

Remark 2.1. If the metric defining the scalar product is the identity matrix, $\mathcal{M}=I_n$, then we get the standard Euclidean space $(\mathbb{R}^n, I_n)$ supplied with the natural dot product: $\langle .,.\rangle$. We denote by $\|\cdot\|_2$ the natural Euclidean norm.

It is then possible to define volumes and angles in an Euclidean metric space. Let K be a bounded subset of $\mathbb{R}^n$, the *volume* of K in metric $\mathcal{M}$ is

$$|K|_{\mathcal{M}} = \int_K \sqrt{\det(\mathcal{M})}\,\mathrm{d}K = \sqrt{\det(\mathcal{M})}|K|_{I_n}.$$

The *angle* between two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by the unique real $\theta\in[0,\pi]$ such that

$$\cos(\theta) = \frac{\langle \mathbf{u},\mathbf{v}\rangle_{\mathcal{M}}}{\|\mathbf{u}\|_{\mathcal{M}}\|\mathbf{v}\|_{\mathcal{M}}}.$$

We now consider the more general case where the metric, and thus the scalar product, vary all over the domain.

Definition 2.2. A *Riemannian metric space* is a continuous manifold $\Omega\subset\mathbb{R}^n$ supplied with a smooth metric $\mathcal{M}(\cdot)$. We denote it by $(\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\Omega}$. The restriction of the metric to a point $\mathbf{x}$ of the manifold defines a scalar product on the tangent space $T_{\mathbf{x}}\Omega$. The tangent space equipped with this structure is an Euclidean metric space.

Contrary to the Euclidean metric space case, the distance between two points, i.e., the shortest path, is no more the straight line but it is given by a geodesic. Nevertheless, in the context of mesh generation or mesh adaptation, we are not interested in the distance between two points but in the length of a path given by an edge of the mesh. More precisely, in a Riemannian metric space $(\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\Omega}$, the *length* of an edge $\mathbf{pq}$ is calculated by using the straight line parametrization $\gamma(t)=\mathbf{p}+t\mathbf{pq}$, $t\in[0,1]$:

$$\ell_{\mathcal{M}}(\mathbf{pq}) = \int_0^1 \|\gamma(t)\|\,\mathrm{d}t = \int_0^1 \sqrt{{}^t\mathbf{pq}\mathcal{M}(\mathbf{p}+t\mathbf{pq})\mathbf{pq}}\,\mathrm{d}t. \tag{1}$$

Fig. 2 depicts iso-values of segment length from the origin, from the standard Euclidean space to a Riemannian metric space. The plotted function is $f(\mathbf{x}) = \ell_{\mathcal{M}}(\mathbf{ox})$ where $\mathbf{o}$ is the origin of the plane.

2.2. Anisotropic mesh generation

We briefly detail our approach to anisotropic mesh generation based on the notion of *unit mesh*. The generation of anisotropic meshes requires the specification of a mesh size in each direction at each point of the domain. To this end, we consider metric tensors to specify different sizes in different directions. Then, the idea is to work in an adequate Riemannian metric space specifying sizes

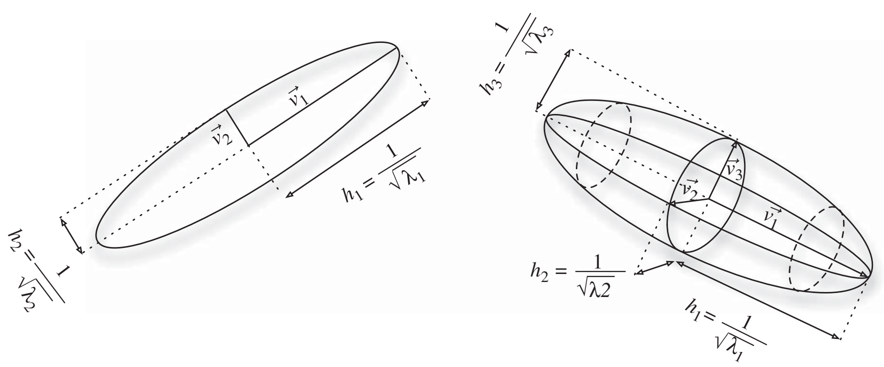

Fig. 1. Ellipse and ellipsoid representing a metric.

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