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Spherical-wave based triangular finite element models for axial symmetric Helmholtz problems

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ABSTRACT

In this paper, six-node hybrid triangular finite element models are devised for axial symmetric Helmholtz problems. In the formulation, boundary and domain approximations to the Helmholtz field are defined for each element. While the boundary approximation is constructed by nodal interpolation, the domain approximation satisfies the Helmholtz equation and is composed of spherical waves with source points located along the axis of symmetry. To formulate rank sufficient six-node elements, a minimal of six wave modes from three source points are required. Two methods of selecting the source points are attempted. In the first method, the directions of the waves passing through the element are essentially parallel to the three lines connecting the parametric center of the element and its three corner (or side) nodes. In the second method, the directions are essentially equally spaced at $2\pi/3$ interval in the r–z-plane. For the attempted examples, the average error ratios of the proposed elements and the conventional element are around 50%.

1. Introduction

In the hybrid finite element method for stress/structural analyses, the displacement-based finite element models are enhanced by introducing stress, strain or another displacement as the additional field variable(s) to the displacement approximation constructed by nodal interpolation [1–10]. In the case of the hybrid-displacement method, the additional field is a domain displacement, which leads to equilibrating stress and may also satisfy some homogeneous boundary conditions [2,3,9,10]. This category of hybrid elements are also known as hybrid-Trefftz or Trefftz elements linked by the displacement-frame or the boundary displacement [4,6–10]. The underlying reason is that the domain displacement is mainly truncated from a Trefftz solution set which is the basis of the Trefftz non-singular boundary element methods.

A major challenge in finite element analyses of Helmholtz problems is that the solutions are spatially oscillating throughout the entire problem domains. While considerable computational saving can be realized by using graded meshes in stress analyses, the practice is not applicable to Helmholtz problems. Hence, the mesh requirement induces tremendous computing load when the wavenumber or the problem domain size increases. To better tackle the issue, a number of wave-based approaches that make use of solution sets for the wave or Helmholtz equations have been proposed in the last decades. These include the Trefftz methods [11–18], the plane-wave basis method [19–22] and the discontinuous enrichment method [23,24], among others.

Though a number of Trefftz boundary element methods have been formulated for Helmholtz problems [11–15], Trefftz finite element models do not appear to be abundant. Among them, the least-square models [16,17] and the traction-frame models [18] can be noted. All Trefftz models possess their own domain approximations which are extracted from Trefftz solution sets. In the plane-wave basis method, the plane wave solutions are employed as the nodal enrichment functions in the context of the partition of unity finite element method [19–22]. The value of the Helmholtz variable at a node is the sum of plane wave solutions which represent plane waves propagating along different directions. Within the element, the Helmholtz variable is obtained by the conventional nodal interpolation. Thus, the system equation unknowns are the amplitudes of the plane waves at the nodes but not the nodal value of the Helmholtz variable. In the discontinuous enrichment method, the coarse scale approximation constructed by the conventional nodal interpolation is enriched by plane wave solutions. The enrichment that is intended to resolve the fine scale phenomenon induces discontinuity across the inter-element boundary [23,24]. Weak enforcement of the continuity is implemented through Lagrange multipliers. While the fine scale enrichments can be condensed at element level, the multipliers which link the enrichments of adjacent elements enter the global equation.

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In this paper, hybrid-Trefftz six-node triangular elements will be formulated for the axial symmetric Helmholtz problem for which there are hardly any advanced finite element models. Unlike the previous Trefftz finite elements, the present ones can readily be incorporated into the standard finite element program framework. Independent boundary and domain approximations to the Helmholtz field are defined. The boundary approximation is constructed by nodal interpolation. Equality of the two approximations is enforced along the element boundary [25–27]. Indeed, the hybrid variational functional employed in the formulation is similar to the functional used in elasticity hybrid-Trefftz elements with displacement-frame [2–4,8–10,18]. The spherical wave solutions are employed to construct the domain approximation. For rank sufficiency, a six-node element has to be equipped with at least six wave modes from three source points. Two methods of selecting the source points are attempted. In the first method, the directions of the waves passing through the element are essentially parallel to the three lines connecting the parametric center of the element and its three corner (or side) nodes. In the second method, the directions are essentially equally spaced at $2\pi/3$ interval. For the attempted examples, the average error ratios of the proposed elements and the conventional element are around 50% at considerably dense meshes.

2. Conventional formulation

Helmholtz equation is often introduced by using the steady state acoustics. The Helmholtz variable u can be the spatial amplitude of the acoustic pressure or the velocity potential. This paper will restrict itself to bounded domains. Under the axial symmetry, a problem domain Ω is often considered thru its cross-sectional area A in the r–z-plane where $r \geq 0$, see Fig. 1. When A is discretized into sub-areas or finite elements A^es, the problem can be summarized as follows:

(a) *Helmholtz equation*: $\nabla^2 u + k^2 u = 0$ in all A^es.
(b) *Natural boundary condition*: $\widehat{\mathbf{n}}^T \nabla^\frown u = \bar{t}$ on all Γ_n^e.
(c) *Essential boundary condition*: $u = \bar{u}$ and $\delta u = 0$ on all Γ_u^e.
(d) *Natural interfacial condition*: $(\widehat{\mathbf{n}}^T \nabla^\frown u)^+ + (\widehat{\mathbf{n}}^T \nabla^\frown u)^- = 0$ on all Γ_m^e.
(e) *Essential interfacial condition*: $u^+ = u^-$ and $\delta u^+ = \delta u^-$ on all Γ_m^e.

In the above expressions, ∇^2 is the Laplace operator (see the appendix), $\nabla^\frown = (\partial/\partial r, \partial/\partial z)^T$, $\widehat{\mathbf{n}} = (\cos\theta_n, \sin\theta_n)^T$ where θ_n is the inclination of the outward normal vector of the element boundary to the r-axis, k is the wavenumber, δ is the variational symbol and Γ_m^e is the inter-element boundary. Moreover, $(\ )^+$ and $(\ )^-$ denote the braced quantities at the two sides of Γ_m^e. In the absence of dissipation, k is real; otherwise it is complex. For simplicity, it will be assumed as usual that element boundary ∂A^e can be partitioned into the non-overlapping portions Γ_t^e, Γ_u^e and Γ_m^e, i.e.

$$\Gamma_t^e \cup \Gamma_u^e \cup \Gamma_m^e = \partial A^e \ \text{ and } \ \Gamma_t^e \cap \Gamma_u^e = \Gamma_u^e \cap \Gamma_m^e = \Gamma_m^e \cap \Gamma_t^e = \text{null}. \tag{1}$$

The terms "natural interfacial condition" and "essential interfacial condition" are not widely used. However, they are indeed the interfacial counterparts of the natural and essential boundary conditions, respectively.

The elemental variational functional for the conventional finite element formulation of the Helmholtz problem is known to be

$$\Pi^e = \frac{1}{2}\int_{\Omega^e} [(\nabla u)^T \nabla u - k^2 u^2]\, d\Omega - \int_{S_t^e} \bar{u}_n u\, dS \tag{2}$$

where u satisfies the essential boundary and continuity conditions. Under the axial symmetry, the differential volume $d\Omega$ and differential surface area dS can be replaced by, respectively, $2\pi r\, dA$ and $2\pi r\, d\Gamma$ in which the common factor 2π can be neglected for simplicity. Then, the functional becomes

$$\Pi^e = \frac{1}{2}\int_{A^e} [(\nabla u)^T \nabla u - k^2 u^2] r\, dA - \int_{\Gamma_t^e} \bar{u}_n u r\, d\Gamma \tag{3}$$

For any smooth axial symmetric functions $f = f(r,z)$ and $h = h(r,z)$, the divergence theorem can be read (see the appendix) as

$$\int_{A^e} [(\nabla h)^T (\nabla f) + h \nabla^2 f] r\, dA = \oint_{\partial A^e} h (\widehat{\mathbf{n}}^T \widehat{\nabla} f) r\, d\Gamma \tag{4}$$

With h and f taken to be, respectively, δu and u and recalling that $\delta u = 0$ on Γ_u^e, variation of (3) is

$$\delta \Pi^e = -\int_{A^e} \delta u (\nabla^2 u + k^2 u) r\, dA + \int_{\Gamma_t^e} \delta u (\widehat{\mathbf{n}}^T \widehat{\nabla} u - \bar{u}_n) r\, dS + \int_{\Gamma_m^e} \delta u (\widehat{\mathbf{n}}^T \widehat{\nabla} u) r\, dS \tag{5}$$

It can be seen that the first integral enforces (a), the second integral enforces (b) and the third integral, when $\delta \Pi^e s$ of the adjacent elements are assembled, enforces (d).

Fig. 2(a) shows the six-node triangular element in the global r–z-plane. To formulate a conventional element, r, z and u are obtained by interpolation which can be expressed as

$$r = \sum_{i=1}^{6} N_i r_i, \ z = \sum_{i=1}^{6} N_i z_i \quad \text{and} \quad u = \sum_{i=1}^{6} N_i u_i = [N_1, \ldots, N_6] \begin{Bmatrix} u_1 \\ \vdots \\ u_6 \end{Bmatrix} = \text{Nd} \tag{6}$$

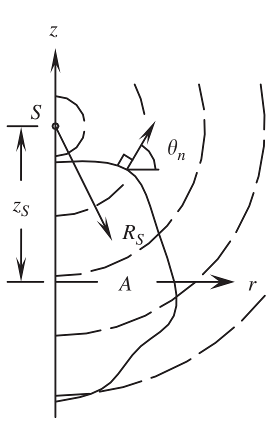

Fig. 1. Cross section A of an axial symmetric body in the r–z-plane. S denotes the source point of the spherical wave $u = \exp(ikR_S)/(kR_S)$ where $R_S^2 = r^2 + (z - z_S)^2$. Over the boundary of A, θ_n denotes the inclination of the outward normal vector to the r-axis.

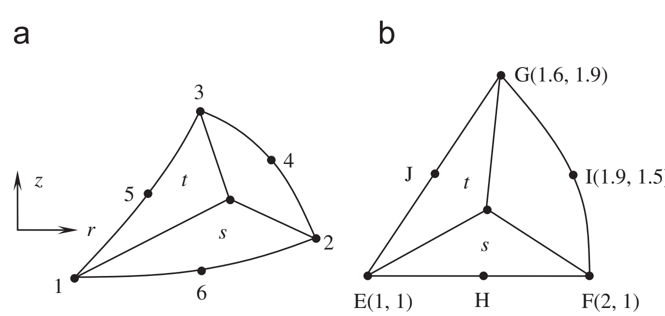

Fig. 2. The six-node triangular element: (a) s and $t \in [0,1]$ are the area coordinates and (b) the single for examining the condition numbers and invariance of the element matrices.