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Modeling of inelastic behavior of curved members with a mixed formulation beam element

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ABSTRACT

The curved beam element in this paper is based on Hu-Washizu variational principle. The nonlinear response of the element arises from the integration of stress–strain relations over several control sections along the element length. The finite element approximation for the beam uses shape functions for stress resultants that satisfy equilibrium and discontinuous strains along the beam. No approximation for the beam displacement field is necessary in the formulation. The proposed element is free from membrane and shear locking. Examples verify the superior performance of the model under linear and nonlinear material conditions.

1. Introduction

The fundamental problem with the development of curved beam finite elements in research has been the locking phenomenon and the accuracy of a developed model under linear and nonlinear material conditions. While the use of displacement-based formulations in straight elements only resulted in shear-locking problem, the geometry of the curved elements exacerbated the situation with the addition of membrane locking.

The early attempts to solve the locking problem resulted in the use of higher order displacement interpolation functions or reduced integration method. Although these methods solved the locking problem, they needed discretization of the member into several elements in order to obtain accurate solutions even under linear elastic material conditions. Such curved beam elements were developed by Babu and Prathap [1], Tessler and Spiridigliozzi [2], Shi and Voyiadjis [3], and Raveendranath et al. [4].

It was realized that the use of differential equations of equilibrium can solve the locking problem. Such elements for straight shear deformable beam elements were proposed by Friedman and Kosmatka [5] and Reddy [6]. The same approach was extended to the curved beam geometry by Friedman and Kosmatka [7], and an accurate 2-noded curved beam element free from locking problem was developed, where the formulation remained in the linear elastic range. This element yielded exact results for displacements with single element discretization. The element formulation based on higher order shape functions that satisfy the homogeneous form of the partial differential equations of equilibrium under linear elastic material response, thus resulting on shape functions with coefficients depending on the geometry and material properties of the beam.

The solution to the locking problem was also dealt with the use of hybrid/mixed methods. The basic principle of mixed methods is the inclusion, besides the displacements, of additional free variables such as stress and/or strain in the variational form. By keeping these secondary variables internal to the element through static condensation, these methods can be implemented in the framework of a general purpose finite element analysis program that is based on the direct stiffness method of analysis.

Mixed methods are more widely used in plate and shell finite elements than beam elements, where independent stress interpolations are mostly used besides the displacements. An application of mixed methods in curved beams by Lee and Sin [8] used independent curvature interpolations along the beam in order to represent the bending energy accurately. Benedetti and Tralli [9] used Hellinger–Reissner principle with interpolation functions for stress resultants derived from the differential equations of equilibrium and standard linear Lagrange shape functions for displacements along the beam. Balasubramanian and Prathap [10] used independent nodal forces besides the displacements, thus increasing the total number of degrees of freedom used in the element formulation. Zhang and Di [11] developed a curved beam element based on Hellinger–Reissner variational principle and by using cubic displacement and quadratic stress interpolations. These hybrid/mixed elements although resulted in locking free performance, required discretization of the member in order to obtain accurate results under linear elastic material

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conditions. Furthermore, nonlinear material response was not investigated in these models.

Despite the accuracy problems faced in curved beam elements with mixed formulation (MF), their variational bases resembled the so-called force formulation elements developed by Ciampi and Carlesimo [12] and Spacone et al. [13]. The latter elements were derived for the analysis of straight beams, used Euler–Bernoulli beam kinematic assumptions; thus they did not include shear deformation. Recently, Taylor et al. [14] developed locking free shear deformable straight beam elements based on three-field Hu-Washizu principle.

This study extends the previous work by Taylor et al. [14] to the analysis of shear deformable curved beam elements in plane. To this end a three-field beam formulation is pursued based on the Hu-Washizu principle that is free of "membrane-locking" and "shear-locking" problems and permits the discretization of each member with a single element. The beam element achieves accuracy with as few elements per span as possible for both the linear and nonlinear ranges of material response.

2. Finite element formulation

2.1. Hu-Washizu functional

The mathematical formulation of the element is based on Hu-Washizu functional with three independent fields, and it has the form

$$\Pi_{\mathrm{HW}}(\boldsymbol{\sigma},\boldsymbol{\varepsilon},\mathbf{u}) = \int_\Omega W(\boldsymbol{\varepsilon})\,\mathrm{d}\Omega + \int_\Omega \boldsymbol{\sigma}^{\mathrm{T}}[\boldsymbol{\varepsilon}^{\mathbf{u}} - \boldsymbol{\varepsilon}]\,\mathrm{d}\Omega + \Pi_{\mathrm{ext}} \quad (1)$$

The independent fields are the stress field $\boldsymbol{\sigma}$, the strain field $\boldsymbol{\varepsilon}$, and the displacement field $\mathbf{u}$. $W(\varepsilon)$ is the strain energy function from which stresses are derived by

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}) = \frac{\partial W(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \quad (2)$$

$\boldsymbol{\varepsilon}^{\mathbf{u}}$ is the strain vector that is compatible with the displacements $\mathbf{u}$. Π_{ext} is the potential energy of the external loading comprised of body forces, as well as displacement and traction boundary conditions, and it has the form

$$\Pi_{\mathrm{ext}} = -\int_\Omega \mathbf{u}^{\mathrm{T}}\mathbf{b}\,\mathrm{d}\Omega - \int_{\Gamma_{\mathrm{t}}} \mathbf{u}^{\mathrm{T}}\mathbf{t}^*\,\mathrm{d}\Gamma - \int_{\Gamma_{\mathrm{u}}} \mathbf{t}^{\mathrm{T}}[\mathbf{u} - \mathbf{u}^*]\,\mathrm{d}\Gamma \quad (3)$$

where the traction $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$ is the dot product of the stress tensor with the outward normal $\mathbf{n}$ to the boundary. In (3) a * superscript denotes the imposed values of variables. The external loading is assumed to be conservative so that the work depends only on the final displacement values $\mathbf{u}$. In Eqs. (1) and (3) the domain of the body is denoted by Ω, while the traction and displacement boundaries are Γ_{t} and Γ_{u}, respectively.

2.2. Kinematical approximations for curved beam

The displacement field along the curved beam element in Fig. 1 is given by the Timoshenko beam theory as

$$\mathbf{u}(\varphi, r) = \begin{Bmatrix} \mathrm{u}_\varphi(\varphi, r) \\ \mathrm{u}_r(\varphi, r) \end{Bmatrix} = \begin{Bmatrix} u(\varphi) - y\theta(\varphi) \\ w(\varphi) \end{Bmatrix} \quad (4)$$

where $r = R + y$ and $x = R\varphi$. By using polar coordinates, the displacement derived strains in the radial and circumferential directions are calculated as

$$\varepsilon_\varphi^u = \frac{\mathrm{u}_r}{r} + \frac{1}{r}\frac{\partial \mathrm{u}_\varphi}{\partial \varphi} = \frac{1}{1 + y/R}\left(\frac{w(\varphi)}{R} + \frac{1}{R}\frac{\mathrm{d}u(\varphi)}{\mathrm{d}\varphi} - \frac{y}{R}\frac{\mathrm{d}\theta(\varphi)}{\mathrm{d}\varphi}\right)$$
$$\gamma_{r\varphi}^u = \frac{1}{r}\frac{\partial \mathrm{u}_r}{\partial \varphi} + \frac{\partial \mathrm{u}_\varphi}{\partial r} - \frac{\mathrm{u}_\varphi}{r} = \frac{1}{1 + y/R}\left(\frac{1}{R}\frac{\mathrm{d}w(\varphi)}{\mathrm{d}\varphi} - \theta(\varphi) - \frac{u(\varphi)}{R}\right) \quad (5)$$

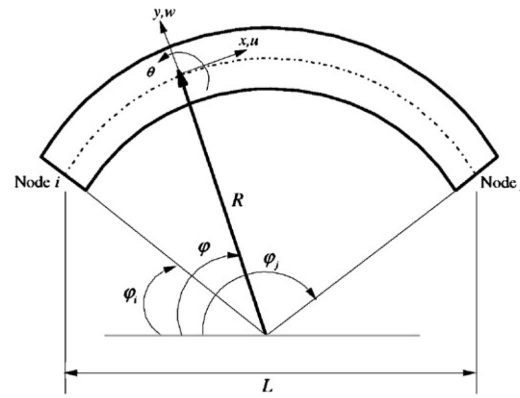

Fig. 1. Description of the curved beam element.

Strains are assumed as independent field, and they are calculated by using the relations in (5)

$$\varepsilon_\varphi = \frac{1}{1 + y/R}(\varepsilon_a(\varphi) - y\kappa(\varphi))$$
$$\gamma_{r\varphi} = \frac{1}{1 + y/R}\gamma(\varphi) \quad (6)$$

where ε_a is the axial deformation, κ is the curvature, and γ is the shear deformation of a section.

2.3. Variation of Hu-Washizu functional

The variation of the Hu-Washizu functional in (1) results in

$$\delta\Pi_{HW} = \int_\Omega \hat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon})\delta\boldsymbol{\varepsilon}\,\mathrm{d}\Omega + \int_\Omega \delta\boldsymbol{\sigma}^{\mathrm{T}}[\boldsymbol{\varepsilon}^{\mathrm{u}} - \boldsymbol{\varepsilon}]\,\mathrm{d}\Omega + \int_\Omega \boldsymbol{\sigma}^{\mathrm{T}}[\delta\boldsymbol{\varepsilon}^u - \delta\boldsymbol{\varepsilon}]\,\mathrm{d}\Omega + \delta\Pi_{\mathrm{ext}} \quad (7)$$

Although this variation is strictly based on a Cauchy elastic material model that possesses a strain-energy function according to (2), it is common to generalize the functional with the substitution of the stress tensor in (7) by $\hat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon})$ in order to include inelastic material response.

In (7) an infinitesimal domain of the beam element with a width of b at position y is equal to

$$\mathrm{d}\Omega = br\,\mathrm{d}\varphi\,\mathrm{d}r = (R + y)b\,\mathrm{d}y\,\mathrm{d}\varphi \quad (8)$$

Using (8) the substitution of the strain variations from (5) and (6) into (7) gives

$$\begin{aligned}\delta\Pi_{HW} = &\int_\Omega \{[\hat{\sigma}_\varphi(\delta\varepsilon_a(\varphi) - y\delta\kappa(\varphi)) + \hat{\sigma}_{r\varphi}\delta\gamma(\varphi)]\}b\,\mathrm{d}y\,R\,\mathrm{d}\varphi \\ &+ \int_\Omega \left\{\delta\sigma_\varphi\left[\left(\frac{w(\varphi)}{R} + \frac{1}{R}\frac{\mathrm{d}u(\varphi)}{\mathrm{d}\varphi} - \varepsilon_a(\varphi)\right)\right.\right. \\ &\left.\left. - y\left(\frac{1}{R}\frac{\mathrm{d}\theta(\varphi)}{\mathrm{d}\varphi} - \kappa(\varphi)\right)\right]\right\} b\,\mathrm{d}y\,R\,\mathrm{d}\varphi \\ &+ \int_\Omega \left\{\delta\sigma_{r\varphi}\left[\frac{1}{R}\frac{\mathrm{d}w(\varphi)}{\mathrm{d}\varphi} - \theta(\varphi) - \frac{u(\varphi)}{R} - \gamma(\varphi)\right]\right\} b\,\mathrm{d}y\,R\,\mathrm{d}\varphi \\ &+ \int_\Omega \left\{\sigma_\varphi\left[\left(\frac{\delta w(\varphi)}{R} + \frac{1}{R}\frac{\mathrm{d}\delta u(\varphi)}{\mathrm{d}\varphi} - \delta\varepsilon_a(\varphi)\right)\right.\right. \\ &\left.\left. - y\left(\frac{1}{R}\frac{\mathrm{d}\delta\theta(\varphi)}{\mathrm{d}\varphi} - \delta\kappa(\varphi)\right)\right]\right\} b\,\mathrm{d}y\,R\,\mathrm{d}\varphi \\ &+ \int_\Omega \left\{\sigma_{r\varphi}\left(\frac{1}{R}\frac{\mathrm{d}\delta w(\varphi)}{\mathrm{d}\varphi} - \delta\theta(\varphi) - \frac{\delta u(\varphi)}{R} - \delta\gamma(\varphi)\right)\right\} \\ &\times b\,\mathrm{d}y\,R\,\mathrm{d}\varphi + \delta\Pi_{\mathrm{ext}}\end{aligned} \quad (9)$$