Finite Elements in Analysis and Design 47 (2011) 73–77

Contents lists available at ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.com/locate/finel

A nodal position finite element method for plane elastic problems

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ARTICLE INFO

Article history:
Received 13 January 2010
Received in revised form
30 July 2010
Accepted 11 September 2010
Available online 30 October 2010

Keywords:
Finite element method
Plane elastic
Dynamic
Modeling
Node position method

ABSTRACT

This paper presents a new nodal position finite element method (NPFEM) as an alternative to the existing finite element method (FEM) for plane elastic problem. The newly developed method addresses the complications of the existing FEM in dealing with dynamic problems experiencing large rigid-body motion coupled with small elastic deformation. Unlike the existing FEM that is based on nodal displacements, the new NPFEM uses nodal positions as basic variables to eliminate the need to decouple the elastic deformation from the rigid-body motion. As a result, it can avoid the accumulated errors arising from the existing FEM over a long period of time by comparing the deformed element with its undeformed status directly. This will be very useful in dynamic modeling of mechanical system where the current positions of parts are more meaningful to designers than the displacements. In addition, the new NPFEM has the potential to address the need in bridging the position based molecular dynamics (MD) to the displacement based finite element (FE) modeling in the multiscale MD/FE analysis. Thus, the NPFEM can provide a unified description in multiscale MD/FE modeling in future.

1. Introduction

Many mechanical systems experience very large rigid-body motions coupled with very small elastic deformations. For instance, the elastic pendulum [1], the towed cable/array systems [2], and the flexible linkage systems [3], just to name a few. For this type of mechanical systems, the current positions of the systems are usually more meaningful than the displacements for the designers and analysts. Unfortunately, the existing finite element methods are displacement based solution procedures. They solve for the displacements of the systems relative to their previous positions in order to obtain the current positions of the systems by adding the displacements to the previous positions. However, this approach suffers from the accumulated errors arising from each step, which will eventually lead to erroneous and unstable solutions over a long period of time due to the violation of energy conservation. For instance, let us consider a rectangular plate experiencing a rigid-body rotation as shown in Fig. 1. The displacements of any point in the plate can be expressed as

$$\begin{cases} u = x(\cos\theta - 1) - y\sin\theta \\ v = x\sin\theta + y(\cos\theta - 1) \end{cases} \tag{1}$$

Accordingly, the Green–Lagrangian strain and the strain energy in the plate should be zero such as

$$\boldsymbol{\varepsilon} = \boldsymbol{e} + \boldsymbol{\eta} = \boldsymbol{0} \quad \text{and} \quad U = \frac{1}{2}\int \boldsymbol{\varepsilon}\boldsymbol{\sigma}\, dV = 0 \tag{2}$$

where

$$\boldsymbol{e} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \end{Bmatrix} = \begin{Bmatrix} \cos\theta - 1 \\ \cos\theta - 1 \\ 0 \end{Bmatrix} \quad \text{and}$$

$$\boldsymbol{\eta} = \frac{1}{2}\begin{Bmatrix} \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 \\ \left(\dfrac{\partial v}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial y}\right)^2 \\ \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} (\cos\theta - 1)^2 + (\sin\theta)^2 \\ (\cos\theta - 1)^2 + (\sin\theta)^2 \\ 0 \end{Bmatrix} \tag{3}$$

The common approach used in the existing finite element method is, if the elastic deformation is small and the material obeys Hooke's law [4]

$$U = \frac{1}{2}\int \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV = \frac{1}{2}\int (\boldsymbol{e}^T \mathrm{De} + \boldsymbol{2e}^T D\boldsymbol{\eta} + \boldsymbol{\eta}^T D\boldsymbol{\eta})\, dV$$
$$\approx \frac{1}{2}\int (\boldsymbol{e}^T \mathrm{De} + \boldsymbol{2e}^T D\boldsymbol{\eta})\, dV \tag{4}$$

where the higher order term $\boldsymbol{\eta}^T D\boldsymbol{\eta}$ is small and ignored. In addition, the $\boldsymbol{D}$ is the elastic matrix

$$\boldsymbol{D} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{12} & d_{11} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \tag{5}$$

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doi:10.1016/j.finel.2010.09.008

Substituting Eqs. (3) and (5) into Eq. (4) leads to

$$U \approx \frac{1}{2}\int (\boldsymbol{e}^T \mathrm{De} + \mathbf{2}\boldsymbol{e}^T D\boldsymbol{\eta})\, dV = -\frac{1}{2}\int (\cos\theta - 1)^2 (d_{11} + d_{12})\, dV \neq 0 \tag{6}$$

The approximated strain energy becomes zero only if the rigid-body rotation θ is small, i.e., $\cos\theta - 1 \approx 1 - 1 = 0$. Thus, the existing approach requires the rigid-body rotation be small within each step. In addition, it introduces an approximation error in each step although it is small. Over a long period of time, the accumulated error may become significant.

Many efforts have been devoted to this problem in the literature, e.g., the symplectic numerical integrator to ensure the energy conservation of the discretized system [5]. These methods are usually complicated in mathematics. Different from the efforts that enhances the existing displacement based finite element methods, some efforts have been devoted to develop an alternative finite element procedure to solve the positions of a system directly after realizing that the positions are the main interest for certain applications. For instance, the author has published a new finite element procedure [2] in 1998 to solve the positions of a towed cable directly. In the same time, Shabana [3] published in 1998 an absolute nodal coordinate finite element method (ANCFEM) to solve the positions and the slopes of a beam directly. Different from the existing finite element method, the ANCFEM represents the rotation angles of a beam with a set of slopes of beam's neutral axis. It solves the difficulty associated with 3D large rotation with added numbers of degrees of freedom. As a result, ANCFEM cannot use the existing beam elements and has to derive its own element. The method has been well accepted in finite element community and later expanded to model the cables and plates/shells [8]. Recently, we standardized our finite element procedure in Ref. [2] and named it as nodal position finite element method (NPFEM) [6,7] for the dynamic modeling of cable systems. Although the name of our method is similar to the existing ANCFEM and both methods solve for the nodal position of an element directly, the NPFEM uses existing cable or bar elements in its procedure, which is different from the ANCFEM. In addition to the nodal position based methods for cables, beams and plates/shells, the authors are unaware of any effort in the literature to solve the plane elastic problem using the nodal position finite element method.

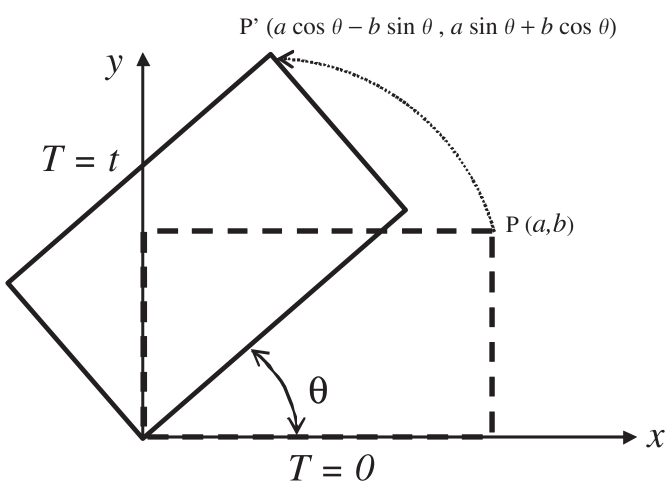

Fig. 1. A plate experiences a rigid-body rotation.

The current paper devotes to the development of the NPFEM for a plane elastic problem. The proposed NPFEM shall be able to use the existing plane elements so that it can be easily integrated into existing finite element codes. The paper contains four sections. Following this introductory section, Section 2 provides a detailed account of the newly developed nodal position finite element method. In Section 3, we validate the newly developed NPFEM by various static and dynamic examples. Finally, in Section 4, we conclude the paper.

2. Nodal position finite element method

Consider a rectangular plane element as shown in Fig. 2 in the global coordinates OXY. The nodal coordinates are denoted as (X_i, Y_i, $i=1, \ldots, 4$). We assume the element moves to a new position under external loads with new nodal coordinates $(\tilde{X}_i, \tilde{Y}_i)$. The local coordinate system of the element (x, y) is defined with x-axis along one side of the element and y-axis perpendicular to the x-axis, see Fig. 2. Denote the new nodal coordinates in the local coordinates as $\tilde{\boldsymbol{x}}_{\boldsymbol{e}} = \{\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \tilde{x}_3, \tilde{y}_3, \tilde{x}_4, \tilde{y}_4\}^T$. To calculate the strain of the deformed element, we add an imaginary undeformed element as a reference. Thus, by inspection, the nodal coordinates of the stress-free imaginary element in the local coordinates can be constructed as $\boldsymbol{x}_{\boldsymbol{e}} = \{\tilde{x}_1, \tilde{y}_1, \tilde{x}_1 + a, \tilde{y}_1, \tilde{x}_1 + a, \tilde{y}_1 + b, \tilde{x}_1, \tilde{y}_1 + b\}^T$. Furthermore, assume the position vectors of an arbitrary point P inside the element before and after deformation can be interpolated using bi-linear shape functions and nodal coordinates, such that

$$\boldsymbol{r} = \{x, y\}^T = \boldsymbol{N}\boldsymbol{x}_{\boldsymbol{e}} \quad \text{and} \quad \tilde{\boldsymbol{r}} = \{\tilde{x}, \tilde{y}\}^T = \boldsymbol{N}\tilde{\boldsymbol{x}}_{\boldsymbol{e}} \tag{7}$$

Accordingly, the elastic displacements in the element can be obtained as follows:

$$\boldsymbol{u} = \{u, v\}^T = \tilde{\boldsymbol{r}} - \boldsymbol{r} = \boldsymbol{N}(\tilde{\boldsymbol{x}}_{\boldsymbol{e}} - \boldsymbol{x}_{\boldsymbol{e}}) \tag{8}$$

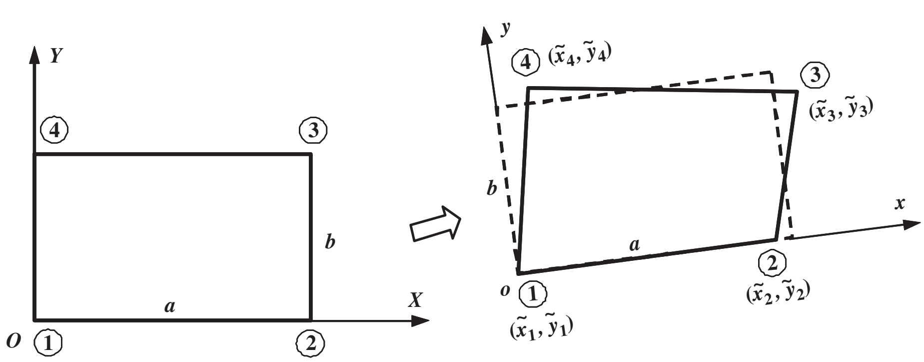

Fig. 2. Element before and after deformation.