**FINITE ELEMENTS
IN ANALYSIS
AND DESIGN**

Optimal priority functions for profile reduction using ant colony optimization

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Abstract

In this paper, nodal ordering for profile reduction is performed using an ant colony optimization. The node priority is expressed as a function of vectors containing the graph connectivity properties. Two methods are proposed for obtaining the priority of nodes to reduce the profile of sparse matrices, which can be viewed as an improvement and extension to Sloan's method. Examples are included to illustrate the performance of the present approaches.
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1. Introduction

Analysis of many problems in structural engineering involves the solution of simultaneous equations arising from the finite element method. Such equations usually involve a sparse coefficient matrix and for large structures 30–70% of the computer execution time may be devoted for solving these equations. Hence some appropriate specified patterns for the coefficients of the corresponding equations have to be provided, like banded form, profile form and partitioned form. These patterns are obtained by appropriate nodal ordering of their models. The frontal methodology due to Irons [1] and the profile method described by George [2] as well as the band-matrix technique of Cuthill–McKee [3] are commonly used.

Profile (variable band) scheme of Jennings [4], the so-called skyline scheme by Felippa [5], can be used when a banded sparse matrix of high order has a wide band and a large number of zeroes inside it. There are many algorithms for profile reduction like reverse Cuthill–McKee [3], Souza and Murray [6], Sloan [7], Gibbs–King [8] and Kaveh [9,10] algorithms.

In finite element analysis, in the case of one degree of freedom per node, performing nodal ordering is equivalent to reordering the equations. In more general problem with m

degree of freedom per node, there are m coupled equations produced for each node. In this case re-sequencing is usually performed on the nodal numbering of the graph model to reduce the bandwidth, profile or wavefront, because the size of this problem is m fold smaller than that for m degree of freedom numbering.

Ant colony, as a metaheuristic, has been used as a combinatorial optimization tool in the last decade. Ant colony algorithms study models which are derived from the observation of real ants' behaviour. This method is one of the random methods for combinatorial optimization problems, where by adding the modified solution in each step, one moves towards the final solution of the problem. The first ant colony optimization (ACO) algorithm used in 1992 by Dorigo [11] was known as ant system (AS). Some modifications on this metaheuristic in recent years made it a suitable method for solving combinatorial optimization problems.

In this article, the mathematical model of an FEM is considered as an element clique graph [12], and nodal ordering is performed using concepts from combinatorial graph theory for finding suitable priority function, as a combination of different graph properties [10]. The coefficients corresponding to two vectors and five vectors of such functions are specified using an ant colony algorithm. The two-vector function can be considered as a modified Sloan method, and the five-vector function may be viewed as an extension to Sloan's algorithm.

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2. Definitions from graphs and sparse matrices

Let $G(N, A)$ be a graph with edges (arches) set A and nodes (vertices) set N with a relation of incidence. The *degree* of a node is the number of edges, incident with the node and 1-*weighted degree* of a node is defined as the sum of the degrees of its adjacent nodes. A *spanning tree* is a tree containing all the nodes of S. A *shortest root tree* (SRT_{n_0}) rooted from a specified node (starting node) n_0 is a spanning tree for which the distance between every node n_j of S and n_0 is minimum, where the *distance* between two nodes is taken as the number of edges in the shortest path between these nodes. A *contour* $C_k^{n_0}$ contains all the nodes with equi-distance k from node n_0. The number of contours of an SRT_{n_0} is known as its *depth*, denoted by $d(SRT_{n0})$, and the highest number of nodes in a contour specifies the *width* of the SRT_{n0}. For the purpose of finding an optimal nodal ordering, one should assign the set of integers $\{1, 2, 3, \ldots, n\}$ to the nodes of G in a suitable manner. Let $As(i)$ be the label or the integer assigned to the node i, where each nodes has different label. The profile of the $N \times N$ matrix related to graph G, for this assignment is defined as

$$P_{As} = \sum_{i=1}^{N} b_i, \tag{1}$$

where the row bandwidth b_i for row i is defined as the number of inclusive entries from the first non-zero element in the row to the $(i+1)$th entry, for this assignment. The efficiency of any given ordering for the profile solution scheme is related to the number of active equations during each step of the factorization process. A row j is defined to be active during elimination of column i if $j \geqslant i$ and there exists $a_{ik} = 0$ with $k \leqslant i$. Hence, at the ith stage of the factorization, the number of active equations is the number of rows of profile that intersect column i, ignoring those rows already eliminated. Let f_i denote the number of equations that are active during the elimination of the variable x_i. It follows from the symmetric structures of the matrix that

$$P_{As} = \sum_{i=1}^{N} f_i = \sum_{i=1}^{N} b_i, \tag{2}$$

where f_i is commonly known as the wavefront or frontwidth. Assuming that N and average value of f_i are reasonably large, it can be shown that a complete profile or front factorization requires approximately $O(NF_{rms}^2)$ operations [13], where $F_{rms}i$ is the root-mean square wavefront, which is defined as

$$F_{rms} = \left[\frac{1}{N} \sum_{i=1}^{N} f_i^2 \right]^{0.5}. \tag{3}$$

The *element clique graph* S of an FEM is a graph whose vertices are the same as those of the FEM, and two vertices n_i and n_j of S are connected with an edge if n_i and n_j belong to the same element in the FEM.

3. An algorithm based on priority queue for profile reduction

The numbering and control of nodes in a priority queue are carried out through the assignment of status, based on the numbering strategy of King [8]. King's algorithm was generalized by Sloan [7], by introducing a priority queue to control the order to be followed in the numbering of the nodes. This algorithm consists of two phases:

Phase 1: Selecting a pair of pseudo-peripheral nodes.

Phase 2: Nodal numbering.

Phase 1 selects a pair of nodes as starting and ending nodes according to the following steps:

Step 1: Choose an arbitrary node s of minimum degree.

Step 2: Generate an $SRT_s = \{C_1^s, C_2^s, \ldots, C_d^s\}$ rooted from s. Let S be the list of the nodes of C_d^s, which is stored in the order of increasing degree.

Step 3: Decompose S into subsets S_j of cardinality $|S_j|$, $j = 1, 2, \ldots, \Delta$, where Δ is the maximum degree of any node of S, such that all nodes of S_j have degree j. Generate an SRT from each node y of S, for the first $1 \leqslant m_j \leqslant \Delta$. If $d(SRT_y) > d(SRT_s)$ then set $s = y$ and go to Step 2.

Step 4: Let e be the root of the longest SRT that has the smallest width. When the algorithm terminates, s and e are the end points of a pseudo-diameter.

Phase 2 reorders the nodes of an element clique graph and ensures that the position of a node in this reordering phase follows a priority rule according to the following steps:

Step 1: Find the status of all nodes. A node can be in the following four states as shown in Fig. 1. A node which has been assigned a new label is defined as *post-active*. Nodes which are adjacent to a post-active node, but do not have a post-active status, are said to be *active*. Each node which is adjacent to an active node, but is not post-active or active, is said *pre-active*. The nodes which are not post-active, active or pre-active, are said to be *inactive*.

Step 2: Prepare the list of the candidate nodes for labelling in the next step, which consists of active and pre-active nodes.

Step 3: Calculate the priority number for all the candidate nodes. For node i the number is obtained from the following relationship:

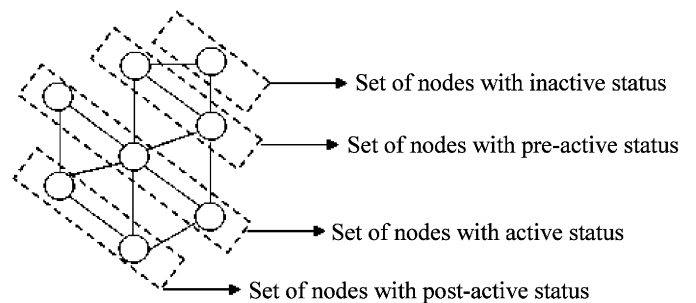$$P_i = W_1 \times \delta_i - W_2 \times D_i, \tag{4}$$

Fig. 1. Nodes in different status.